

# 9 Differentiation

## 9.1 Differentiating $\sin x$ and $\cos x$

1 Find  $\frac{dy}{dx}$  given that:

a  $y = 6 \sin x$     b  $y = -3 \sin \frac{1}{2}x$     c  $y = \frac{1}{3} \sin 5x$   
 d  $y = \cos 4x$     e  $y = 5 \cos 6x$     f  $y = -\frac{1}{2} \cos 7x$

**Hint**

If  $y = \sin kx$ , then  $\frac{dy}{dx} = k \cos kx$

If  $y = \cos kx$ , then  $\frac{dy}{dx} = -k \sin kx$

2 Find  $f'(x)$  given that:

a  $f(x) = 4 \sin x - 3 \cos x$   
 b  $f(x) = \frac{1}{2} \sin 2x + 4 \cos x - 6 \sin \frac{1}{4}x$   
 c  $f(x) = 5x^{\frac{3}{2}} - \frac{4}{x^2} - 6 \sin \frac{1}{3}x$     d  $f(x) = \frac{8}{\sqrt{x}} + 3 \cos \frac{1}{6}x$

**Hint**

Differentiate each term separately.

3  $f(x) = 5 \sin x - 3 \cos x$ . Find the  $x$ -coordinates of the stationary points on the curve with equation  $y = f(x)$  in the interval  $-180^\circ \leq x \leq 180^\circ$ .

**Hint**

Stationary points occur when the gradient is equal to zero.

4  $f(x) = x - \cos 2x$ . Find the equation of the tangent to the curve at the point  $(\frac{\pi}{4}, \frac{\pi}{4})$ .

**Hint**

First differentiate to find the gradient,  $m$ . Then find the equation of the tangent using the equation  $y - y_1 = m(x - x_1)$

← Year 1, Section 12.6

**E** 5  $f(x) = 6 \sin x - 3 \cos 2x$   
 Find the stationary points of the curve  $y = f(x)$  in the interval  $0 \leq x \leq 2\pi$ . (6 marks)

**E/P** 6  $g(x) = 6 \sin 2x - 4 \cos 2x$ . Find the equation of the normal to the curve with equation  $y = g(x)$  at the point  $(\pi, -4)$ . (5 marks)

**E/P** 7  $h(x) = 3 \sin x - 4 \cos x + \frac{1}{2}x^2$ . Show that the equation of the tangent to the curve with equation  $y = h(x)$  at the point with  $x$ -coordinate  $\frac{\pi}{2}$  is  $8y - (32 + 4\pi)x = 24 - 16\pi - \pi^2$ . (6 marks)

**E/P** 8 Prove from first principles that the derivative of  $\cos x$  is  $-\sin x$ . (5 marks)

## 9.2 Differentiating exponentials and logarithms

1 Differentiate:

a  $6e^{-3x}$     b  $5e^{2x} - 7e^{-x}$   
 c  $\frac{7}{e^{5x}}$     d  $\frac{(1 - e^{2x})^2}{e^{2x}}$

**Hint**

If  $y = e^{kx}$ , then  $\frac{dy}{dx} = ke^{kx}$

2 Find  $f'(x)$  given that:

a  $f(x) = \ln 2x$     b  $f(x) = \ln x^2$   
 c  $f(x) = \ln \left(\frac{7}{x}\right)$     d  $f(x) = \ln(4x^3)$

**Hint**

If  $y = \ln x$ , then  $\frac{dy}{dx} = \frac{1}{x}$

First expand the expression using laws of logarithms. ← Year 1, Section 14.5

3 Find  $\frac{dy}{dx}$  given that:

a  $y = 3^x$

b  $y = 5^{2x}$

c  $y = \left(\frac{2}{5}\right)^{4x}$

d  $8^{-x}$

**Hint**

If  $y = a^{kx}$ , where  $k$  is a real constant and  $a > 0$ , then  $\frac{dy}{dx} = a^{kx} k \ln a$ .

4 Find the equation of the tangent to the curve  $y = 4^x$  at the point with  $x$ -coordinate  $-\frac{1}{2}$

**(E)** 5 Given  $f(x) = e^{-x} + 2 \ln x^3$ , find  $f'(x)$ . (3 marks)

**(E/P)** 6 A curve has equation  $y = 3e^x - \ln x^2$ . Show that the equation of the normal at the point with  $x$ -coordinate 1 is  $x + (3e - 2)y - 9e^2 + 6e - 1 = 0$ . (6 marks)

**(E/P)** 7  $f(x) = 4^x + 4^{-x}$ . Show that the equation of the tangent to the curve with equation  $y = f(x)$  at the point with  $x$ -coordinate  $\frac{1}{2}$  is  $y = (\ln 8)x + \frac{5}{2} - \frac{1}{2} \ln 8$ . (6 marks)

**(E/P)** 8  $f(x) = 9^{3x} + 3^{6x}$   
 a Demonstrate, clearly showing all your steps, that  $f(x)$  can be written in the form  $a(3^{kx})$  where  $a$  and  $k$  are constants to be found. (2 marks)

b Hence, or otherwise, show that  $f'(x) = (4 \ln 3)(3^{6x+1})$  (3 marks)

### 9.3 The chain rule

1 Find  $\frac{dy}{dx}$  for each of the following:

a  $y = (4 - 3x)^3$

b  $y = (6 + 5x)^{\frac{1}{2}}$

c  $y = (5 + 2x^3)^4$

d  $y = (7 + 3x^2)^{-5}$

e  $y = \frac{6}{\sqrt{4 - 5x}}$

f  $y = \frac{1}{6 - x}$

g  $y = \sqrt[3]{3 + 2x^4}$

h  $y = \sqrt{7x^3 - 4}$

2 Differentiate:

a  $y = \sin^3 x$

b  $y = \ln x^4$

c  $y = e^{\cos 4x}$

d  $y = (\ln x)^2$

e  $y = \ln(\cos x)$

f  $y = \sin(e^{-3x})$

3  $f(x) = (3 - \ln 6x)^2$

a Find  $f'(x)$ .

b Evaluate  $f'\left(\frac{e^2}{6}\right)$ .

**Hint**

If  $y = (f(x))^n$ , then  $\frac{dy}{dx} = n(f(x))^{n-1}f'(x)$ .

**Hint**

You can use the chain rule to differentiate a function of a function.

If  $y = f(g(x))$ , then  $\frac{dy}{dx} = f'(g(x))g'(x)$ .

- 4 Find the value of  $\frac{dy}{dx}$  at the point (3, 6) on the curve with equation  $2y^2 - 4y = x$ .

**Hint**

First find  $\frac{dx}{dy}$   
Then use the fact that  $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

- E/P** 5 A curve has equation  $y = (2x - 3)^5$ . Find the equation of the normal to the curve at the point  $P$  with  $x$ -coordinate 2. (5 marks)

- E/P** 6  $f(x) = \frac{3}{(6-x)^3}$ ,  $x \neq 6$ . Find the tangent to the curve with equation  $y = f(x)$  at the point  $P$  with  $x$ -coordinate 4. Give your answer in the form  $ax + by + c = 0$ . (7 marks)

- E/P** 7  $f(x) = 3\sin^2 x - 2\cos^2 x$ . Show that the equation of the tangent to the curve with equation  $y = f(x)$  at the point  $Q$  with  $x$ -coordinate  $-\frac{5\pi}{6}$  is  $12y - 30\sqrt{3}x - 25\pi\sqrt{3} + 9 = 0$ . (6 marks)

- E/P** 8 The curve  $C$  has equation  $x = -2\sin 4y$ . Show that the equation of the normal to  $C$  at the point  $A$  with  $y$ -coordinate  $\frac{2\pi}{3}$  is  $4x + y + 4\sqrt{3} - \frac{2\pi}{3} = 0$ . (6 marks)

- E/P** 9 Find the equation of the tangent to the curve with equation  $y = 2^{(x^9)}$  at the point with coordinates  $(-1, 2)$ . Give your answer in the form  $ax + by + c = 0$ , where  $a$ ,  $b$  and  $c$  are exact values to be found. (6 marks)

## 9.4 The product rule

- 1 Find  $\frac{dy}{dx}$  for each of the following:

a  $y = x(4 - 5x)^3$

b  $y = 3x^3(2x - 1)^6$

c  $y = \frac{4}{x^2}(6x - 7)^4$

d  $y = 8x^2(2x^3 - 4)^{-3}$

**Hint**

If  $y = uv$ , then  $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$ ,  
where  $u$  and  $v$  are functions of  $x$ .

- 2 Find  $f'(x)$  given that:

a  $f(x) = x^2 \cos 2x$

b  $f(x) = e^{3x} \sin 5x$

c  $f(x) = e^{-x}(2x^2 + 3)^3$

d  $f(x) = -3 \cos 4x \sin 2x$

e  $f(x) = x^2 \ln x$

f  $f(x) = e^{4x} \ln(\cos 3x)$

**Hint**

If  $f(x) = g(x)h(x)$  then  
 $f'(x) = g(x)h'(x) + h(x)g'(x)$

- 3 Find the value of  $\frac{dy}{dx}$  at the point (1, 4) on the curve with equation  $y = 4x^3(1 - 2x^2)^4$ .

**Hint**

Evaluate  $\frac{dy}{dx}$  when  $x = 1$ .

- 4 Differentiate with respect to  $x$ :

a  $(x^4 - 3x^2)e^{-2x}$

b  $(4 \cos 2x - \sin x)e^{3x}$

c  $(x^2 - 5 \cos \frac{1}{2}x) \ln 4x$

- E** 5 Differentiate with respect to  $x$ :

a  $e^{-x} \cos 3x$

b  $x^4 \ln(\sin x)$

(2 marks)

(2 marks)

- E/P** 6 a A curve  $C$  has equation  $y = 2x^4(2 - 5x^2)^3$ . Show that  $\frac{dy}{dx} = x^3(2 - 5x)^n(ax^2 + bx + c)$ , where  $n, a, b$  and  $c$  are constants to be found. (4 marks)
- b Hence find the exact  $x$ -coordinates of the stationary points of  $C$ . (4 marks)
- E/P** 7 The curve  $C$  has equation  $f(x) = e^{4x} \sin x$ .
- a Show that the turning points of  $C$  occur when  $\tan x = -\frac{1}{4}$ . (3 marks)
- b Show that the equation of the normal to  $C$  at the point where  $x = \frac{\pi}{2}$  is  $x + 4e^{2\pi}y - 4e^{4\pi} - \frac{\pi}{2} = 0$ . (3 marks)
- E/P** 8 A curve has equation  $y = (x - \pi) \sin 2x$ . Show that the equation of the tangent to the curve at the point with  $x$ -coordinate  $\frac{3\pi}{4}$  is  $x + y = \pi$ . (6 marks)

## 9.5 The quotient rule

- 1 Find  $f'(x)$  given that:

a  $f(x) = \frac{4x+3}{2x-9}$       b  $f(x) = \frac{6x^2}{3-5x}$

c  $f(x) = \frac{1-8x}{(3x^3+2)^2}$       d  $f(x) = \frac{7x^2-9x}{\sqrt{x^2+6}}$

**Hint**

If  $f(x) = \frac{g(x)}{h(x)}$  then

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$$

- 2 Find  $\frac{dy}{dx}$  for each of the following:

a  $y = \frac{4x^2}{\cos 2x}$       b  $y = \frac{x - e^{-x}}{\ln x}$

c  $y = \frac{e^x}{\cos^2 x}$       d  $y = \frac{5 - e^{-x}}{\sin 4x}$

**Hint**

If  $y = \frac{u}{v}$  then  $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$  where  $u$  and  $v$  are functions of  $x$ .

- 3 Find the gradient of the curve with equation  $y = \frac{8-2x}{4x+1}$  at the point  $(\frac{1}{2}, \frac{7}{3})$
- 4 The curve  $C$  has equation  $y = \frac{x^2-6x}{e^{2x}}$
- a Find  $\frac{dy}{dx}$ .      b Find the equation of the normal to  $C$  at  $(1, -\frac{5}{e^2})$ .
- E** 5 Given  $y = \frac{\sin^2 x}{e^{2x}}$ , find  $\frac{dy}{dx}$ . (4 marks)
- E** 6 Differentiate  $\frac{e^{4x}}{x(x-5)}$  with respect to  $x$ . (4 marks)
- E/P** 7 The curve  $C$  has equation  $x = \frac{\cos 2y}{e^y}$ . Find the equation of the tangent to the curve at the point where  $y = \frac{\pi}{8}$  in the form  $ax + by + c = 0$ . (6 marks)
- E/P** 8 The diagram shows part of the graph with equation  $y = f(x)$ , where  $f(x) = \frac{\cos^2 x}{e^x}$
- a Show that at the stationary points  $A, B, C$  and  $D$ , either  $\cos x = 0$  or  $\tan x = -\frac{1}{2}$ . (4 marks)
- b Hence, or otherwise, find the  $x$ -coordinates of the points  $A, B, C$  and  $D$ , giving your answers in radians to 2 decimal places. (2 marks)

