

Sequences

- 1** Work out the binomial expansions of these expressions up to and including the term in x^3

a $(1+x)^{-3}$	b $(1+x)^{\frac{1}{2}}$
c $(1+x)^{\frac{2}{3}}$	d $(1+4x)^{-1}$
e $(1-3x)^{-2}$	f $\left(1+\frac{1}{2}x\right)^{\frac{1}{3}}$

- 2** Work out the binomial expansions of these expressions up to and including the term in x^2

a $(2+x)^{-4}$	b $(3+2x)^{-1}$
c $(4-3x)^{\frac{3}{2}}$	d $\frac{3}{(3+x)^2}$
e $\sqrt{9+x}$	f $\frac{6}{3-4x}$
g $\frac{8}{\sqrt{4+9x}}$	h $\frac{27}{2(3-2x)^2}$

- 3** Work out the binomial expansions of these expressions, up to and including the term in x^2 . Simplify coefficients in terms of the positive constant k

a $(1+kx)^{-3}$	b $\sqrt{4+kx}$
c $\left(\frac{k}{k+x}\right)^2$	d $(\sqrt{k}+x)^{-2}$

- 4** Evaluate the range of values of x for which the binomial expansion of these expressions is valid.

a $(1+4x)^{-3}$	b $(3+x)^{-2}$
c $(4-5x)^{\frac{3}{2}}$	d $\left(\frac{1}{2}+\frac{1}{3}x\right)^{\frac{1}{4}}$

- 5** Given that $(1+x)^{-2} = 1-2x+3x^2+\dots$, work out the expansion of these expressions up to and including the term in x^2

a $x(1+x)^{-2}$	b $(2+x)(1+x)^{-2}$
c $(3-2x)(1+x)^{-2}$	d $(1-3x^2)(1+x)^{-2}$

- 6** By simplifying these expressions, or otherwise, work out their binomial expansions up to and including the term in x^2

a $\frac{1}{4+4x+x^2}$	b $\frac{4-6x}{4-9x^2}$
c $\frac{3}{\sqrt{1+4x+4x^2}}$	d $\frac{2}{(2-\sqrt{x})} \times \frac{6}{(2+\sqrt{x})}$

- 7** Given that the binomial expansion of $(a+4x)^{-3}$, where $a > 0$, is only valid for $-\frac{1}{2} < x < \frac{1}{2}$

- a** Calculate the value of a
- b** Hence evaluate the coefficient of x^3 in this expansion.

- 1 a** Work out the first three terms, in ascending powers of x , in the binomial expansion of $\sqrt{1+5x}$
- b** Use this series approximation and $x = 0.05$ to show that $\sqrt{5} \approx \frac{143}{64}$
- c** Explain why substituting $x = 0.8$ into the full expansion of $\sqrt{1+5x}$ does not give an answer equal to $\sqrt{5}$
- 2 a** Work out the expansion of $(4+x)^{\frac{1}{2}}$, up to and including the term in x^2 . State the range of values of x for which the full expansion of $(4+x)^{\frac{1}{2}}$ is valid.
- b** Use the series approximation found in part **a** and
- i** $x = 0.25$ to show that $\sqrt{17} \approx \frac{2111}{512}$
- ii** $x = -0.25$ to calculate an estimate for $\sqrt{15}$
- c** Hence show that $\sqrt{17} - \sqrt{15} \approx \frac{1}{4}$
- 3** The first three terms in the binomial expansion of $(1+3x)^n$ are $1+kx-x^2$, where n and k are constants, $n > \frac{1}{2}$
- a** Work out the value of n and the value of k
- b** Use this series approximation and a suitable value of x to calculate an estimate for $\sqrt[3]{1.69}$

- 4** The first three terms in the binomial expansion of $(1+px)^n$ are $1+12x+24x^2$, where p and n are constants.
- Using the series approximation and $x = 0.01$, show that $\sqrt{3} \approx \frac{1403}{810}$
- 5 a** Work out the expansion of $\frac{3+7x}{(1+x)(1+3x)}$ up to and including the term in x^3
- b** State the range of values of x for which this full expansion is valid.
- 6** The curve C has equation $y = f(x)$, where $f(x) = \frac{2-3x}{2-3x-2x^2}$
- a** Work out the expansion of $f(x)$ up to and including the term in x^2
- b** Sketch a graph which approximates C for values of x close to zero.
- 7** $f(x) = \frac{x+4}{(1+x)(2+x)^2}$
- The curve C has equation $y = f(x)$ and passes through point P with x -coordinate -0.1
- a** Calculate the y -coordinate of point P , giving your answer to 1 decimal place.
- b** Work out the first three terms in the expansion of $f(x)$, giving your answer in the form $p+qx+rx^2$, stating the value of each constant p , q and r
- c** Use your answers to parts **a** and **b** to show that the tangent to this curve at point P can be approximated by the line with equation $y = 0.98 - 2.2x$

Challenge

- 8 a** By using $\cos^2 x + \sin^2 x \equiv 1$, show that $\frac{1}{\cos^2 x} = 1 + \sin^2 x + \sin^4 x + \sin^6 x + \dots$ and explain why this expansion is valid for $-90^\circ < x < 90^\circ$

It is given that

$$\frac{1}{\sin^2 x} = 1 + \cos^2 x + \cos^4 x + \cos^6 x + \dots$$

where $0^\circ < x < 90^\circ$

- b** Read through this argument and explain where the error has occurred.

$$\text{For } 0^\circ < x < 90^\circ, \frac{1}{\cos^2 x} \approx 1 + \sin^2 x$$

$$\text{and } \frac{1}{\sin^2 x} \approx 1 + \cos^2 x$$

Adding these expressions

$$\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \approx (1 + \cos^2 x) + (1 + \sin^2 x)$$

$$\text{so } \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \approx 3$$

(as $\cos^2 x + \sin^2 x \equiv 1$)

But when $x = 15^\circ$,

$$\frac{1}{\cos^2 15^\circ} + \frac{1}{\sin^2 15^\circ} = 16$$

which is not close to 3

In these questions, u_n represents the n th term of a sequence where $n \in \mathbb{Z}^+$

- 1** Work out the values of the first four terms of these sequences.
- a** $u_n = 5n + 3$ **b** $u_n = n^2 - 3$
- c** $u_n = 2n^2(n-1)$ **d** $u_n = \frac{n}{n^2 - 6}$
- 2** Work out the values of u_2 , u_3 and u_4 for these sequences.
- a** $u_{n+1} = 2u_n + 5$, $u_1 = 1$
- b** $u_{n+1} = u_n^2 + 2$, $u_1 = 1$
- c** $u_{n+1} = \frac{1}{u_n + 1}$, $u_1 = 3$
- d** $u_{n+1} = (u_n - 3)^2$, $u_1 = 5$
- 3** Describe the list u_1, u_2, u_3, u_4 as either an increasing sequence, a decreasing sequence or neither where
- a** $u_n = \frac{6}{n+1}$ **b** $u_n = n^2 + 4n - 3$
- c** $u_n = n^2 - 6n + 1$ **d** $u_n = \sqrt[n]{n}$

- e** $u_n = \log_{n+1}(n+2)$
- f** $u_n = \sin(30n)^\circ \cos(30n)^\circ$
- 4** Evaluate the order of these periodic sequences.
- a** $u_n = \tan(n \times 60^\circ)$
- b** $u_{n+1} = 3 - \frac{9}{u_n}, u_1 = 1$
- c** $u_{n+1} = (-1)^n u_n, u_1 = 2$
- d** $u_{n+1} = \frac{2}{9} \left(3 - \frac{1}{u_n} \right), u_1 = 3$
- 5** The n th term of a sequence is given by $u_n = 3n - 2$
- a** Calculate the value of n such that $u_n = 229$
- b** Work out the values of n for which $u_n > 1000$
- 6** The n th term of a sequence is given by $u_n = n^2 - 8n + 18$
- a** Calculate the value of n such that $u_n = 83$
- b** Work out the value of the smallest term in this sequence.
- 7** By solving an equation, find the limit L of these sequences as $n \rightarrow \infty$. Where appropriate, give answers in simplified surd form.
- a** $u_{n+1} = 0.4u_n + 3$ **b** $u_{n+1} = 0.6u_n - 3$
- c** $u_{n+1} = 3 - 0.25u_n$ **d** $u_{n+1} = \frac{3}{5}u_n - \frac{1}{2}$
- e** $u_{n+1} = 0.1(7u_n + 2)$ **f** $u_{n+1} = \frac{\sqrt{5}u_n + 4}{3}$
- g** $u_{n+1} = (\sqrt{3} - 2)u_n + 2\sqrt{3}$
- h** $u_{n+1} = \frac{1}{4}u_n^2 + 1$
- Use your calculator or a spreadsheet with starting value $u_1 = 1$ to verify each answer.

In all questions, assume $n \in \mathbb{Z}^+$

- 1** A sequence is defined by the equation $u_{n+1} = 4u_n + k, u_1 = 1$, where k is a constant. Given that $u_3 = 31$
- a** Work out the value of u_4
- b** Evaluate the sum of the first four terms of this sequence.

- 2** A sequence is defined by the equation $u_{n+1} = au_n + 7, u_1 = -1$, where a is a constant. Given that $u_3 = 19$
- a** Show that $a^2 - 7a + 12 = 0$
- b** Work out the possible values of u_4
- 3** A sequence is defined by $u_{n+1} = au_n + b$ for constants a and b . The first three terms of this sequence are 15, 24 and 42, respectively.
- a** Calculate the values of a and b
- Given that $u_{m+1} = 1.96u_m$, where $m \geq 2$
- b** Work out the value of u_{m+2}
- 4** A periodic sequence satisfies the equation $u_{n+1} = a - \frac{2a}{u_n}$, where $a > 0$ is a constant. The sequence has period 3 and $u_3 = -2$
- a** Show that
- i** $u_2 = a - 1$ **ii** $a^2 - a - 2 = 0$
- b** Hence find the sum of the first 300 terms of this sequence.
- 5** Monthly observations at a wildlife sanctuary suggest that its population of otters is declining. The number of otters, p_n , seen during the $(n+1)$ th observation was modelled by the equation $p_{n+1} = 0.75(p_n + 8)$. During the first observation 56 otters were seen.
- a** Show that there was a 12.5% decrease in the number of otters seen in the third observation compared to the second.
- In one particular observation, at least 24 otters were observed.
- b** Show that the number of otters seen in the next observation was also at least 24
- c** Explain why, according to this model, the population of otters will never die out.

- 6** A code breaking competition consists of 10 rounds, each more difficult than the previous one. A round starts when the code is issued and contestants must break the code within two hours before being allowed to progress to the next round. It takes one of the contestants, Sam, m_n minutes to break the code in round n where $m_{n+1} = a(m_n - 1)$ and a is a positive constant. Sam takes 4 minutes to break the code in round 2 and 10 minutes to break the code in round 4

- Show that $3a^2 - a - 10 = 0$
- Work out the time, in minutes, that it takes Sam to break the code in round 1
- How many rounds of this competition does Sam successfully take part in?

- 7** In preparation to run a race, Paula undertakes weekly training sessions. In the n th session she runs e_n miles due East from her house, turns due South and runs s_n miles and then runs directly back to her house, so that the path she takes in each session is a right-angled triangle.

In the first session she runs 1.5 miles due East and 2 miles due South.

- Calculate the total distance she runs in session 1

For $n \geq 1$, it is given that $e_{n+1} = \frac{2}{3}e_n + 2$ and that $s_{n+1} = \frac{1}{2}s_n + k$ where k is a constant. In session 2, Paula runs 12 miles in total.

- Show that $\sqrt{10 + 2k + k^2} = 8 - k$ and hence evaluate the value of k

Paula correctly calculates that, to the nearest mile, the distance she runs in session 8 equals the length of the race.

- Calculate, to the nearest mile, the length of the race.

- 8** A sequence is defined by $u_{n+1} = pu_n + 6$, $u_1 = 5$, where p is a positive constant.

Given that $u_3 = 9.2$

- Show that $5p^2 + 6p - 3.2 = 0$

The limit of u_n as $n \rightarrow \infty$ is L

- Find the value of L

Challenge

- 9** A sequence is defined by $u_{n+1} = pu_n + q$, $u_1 = 12$, where p and q are constants. The second term of this sequence is 10 and the limit as $n \rightarrow \infty$ is 2
- Find the value of p and the value of q
 - For these values of p and q , find the limit as $n \rightarrow \infty$ of the sequence defined by $v_{n+1} = qv_n + p$, $v_1 = 12$

In these questions u_n represents the n th term of a sequence where $n \geq 1$

- 1** Work out the values of the first four terms of these arithmetic sequences.

a $u_n = 3 + 2n$ **b** $u_n = 12 - 3n$

c $u_n = 7n - 4$ **d** $u_n = \frac{3}{2} + \frac{5}{2}n$

- 2** Work out expressions for the n th terms of these arithmetic sequences, simplifying each answer as far as possible.

a 7, 11, 15, ... **b** 2, -1, -4, ...

c 4, 5.5, 7, ... **d** $2, \frac{8}{3}, \frac{10}{3}, \dots$

e $\frac{3}{5}, 1, \frac{7}{5}, \dots$ **f** $\sqrt{2}, \sqrt{8}, \sqrt{18}, \dots$

- 3** Work out the value of the 50th term in each of these sequences.

a 3, 6, 9, ... **b** 8, 5, 2, ...

c $\frac{5}{4}, \frac{5}{2}, \frac{15}{4}, \dots$ **d** 5.4, 3.2, 1.0, ...

- 4** Evaluate these arithmetic series.

The number of terms in each sum is shown in brackets.

a $60 + 57 + \dots + 18$ (15 terms)

b $10 + 6 + 2 + \dots - 34$ (12 terms)

c $2 + 2.5 + 3 + \dots$ (99 terms)

d $\sqrt{3} + \sqrt{27} + \dots$ (10 terms)

- 5** Work out the number of terms in each of these arithmetic series.

a $4 + \dots + 151$ when $S_n = 3875$

b $7 + \dots + 52$ when $S_n = 295$

c $4 + \dots + 44$ when $S_n = 2400$

d $\frac{1}{5} + \dots + \frac{28}{5}$ when $S_n = 29$

- 7** Quickline trains has invested in new timetabling software. The number of complaints received from their passengers each month in the following year was modelled using an arithmetic sequence. In the first month, 152 complaints were received. This decreased to 140 complaints in the second month.

a Use this model to show that a total of 1032 complaints were received during the year.

The train company claimed that there was roughly a 60% reduction in the number of complaints received between the first six months and the last six months of that year.

b Use this model to determine whether this claim is accurate.

c Comment on the suitability of this model in the following year.

- 8** The number of departures from a new airport each month was modelled using an arithmetic sequence. There were 450 departures in its first month of being operational. In the first three months of the airport being operational, there were 1470 departures.

a Use this model to work out the total number of departures in the first six months of the airport being operational.

If the number of departures in any complete month exceeded 1000 then the airport authority paid an environmental fine of £5000 for that month.

- b** Use this model to calculate how much the airport authority was fined in its first two years of being operational.

- 9** Jim has trained for a new job in sales. Each year he earns a basic salary of £24 000 plus 5% commission on the value of any sales made during that year. In his first working year, Jim earned £27 500. In his second working year, he made sales worth £80 000. The amount Jim earned each year was modelled using an arithmetic sequence.

a Use this model to work out

i The total amount Jim earned in his first five working years,

ii The total value of sales Jim made in his first five working years.

At the end of each working year, Jim paid 8% of the amount earned that year into a pension fund. The amount paid in each year formed an arithmetic sequence.

b Calculate the minimum number of whole years Jim will need to work for these pension contributions to have a total value of more than £30 000

c Give one criticism of using an arithmetic sequence to model the amount Jim earned each year.

Challenge

- 10** Without using a calculator, work out a relationship between the sums

$$1+2+3+\dots+n \text{ and } 1^3+2^3+3^3+\dots+n^3$$

and hence find the value of the positive integer n for which

$$1^3+2^3+3^3+\dots+n^3=90\,000$$

- 1 Work out the values of the first four terms of the geometric sequences defined by

a $u_n = 4 \times 3^{n-1}$ **b** $u_n = 3 \times 4^{n-1}$
c $u_n = 8 \times 0.5^n$ **d** $u_n = 6 \times 3^{-n}$
e $u_n = 5 \times 2^{-n-1}$ **f** $u_n = 2 \times 0.5^{-(n-3)}$

- 2 Work out an expression for the n th term of these geometric sequences.

a 5, 10, 20, ... **b** 36, 24, 16, ...
c 2, -6, 18, ... **d** -8, 6, -4.5, ...
e 7, -24.5, 85.75, ... **f** $1, \frac{1}{6}, \frac{1}{36}, \dots$

- 3 Evaluate these geometric series. Give non-integer answers to 3 significant figures.

a $6 + 12 + 24 + \dots$ (10 terms)
b $64 + 96 + 144 + \dots$ (7 terms)
c $50 + 40 + 32 + \dots$ (15 terms)
d $4 - 6 + 9 + \dots$ (20 terms)

- 4 Use a formula to evaluate these geometric series.

a $\sum_{k=1}^{12} (2 \times 3^k)$ **b** $\sum_{k=1}^6 (192 \times 0.5^k)$
c $\sum_{k=3}^{10} (5 \times 2^k)$ **d** $\sum_{k=5}^{15} 3 \times (-2)^{k-1}$

- 6 At the start of year 1, £2500 is invested in a savings account. The account paid 2% compound interest per year and no further deposits or withdrawals were allowed. The balance (in £) of the account at the start of year n was modelled using a geometric sequence with common ratio r

- a** Write down the value of r

Use this model to calculate, to the nearest £

- b** The balance of the account at the start of year 4
c The amount of interest earned on the account in year 7

- 7 Toni bought a car for £24 000 which decreased in value by 25% each year after its purchase. The value of the car n years after its purchase was modelled by a geometric sequence with common ratio r

- a** Work out the value of r
b Use this model to show that three years after its purchase the value of the car was £10 125

Toni sold the car in the year its value fell below £2000

- c** Use this model to find the number of whole years Toni owned the car.

- 8 In his first year of driving, Tom drove 3125 miles. In his first two years of driving he drove 5625 miles. The distance (in miles) driven in Tom's n th year of driving was modelled using a geometric sequence.

- a** Use this model to

- i** Work out the distance Tom drove in his fourth year of driving,
ii Calculate the total distance Tom drove in his first six years of driving,
iii Show that the total distance Tom can drive in his lifetime is less than 15 625 miles.

- 9 A rectangular piece of foil has length 50 cm, width 40 cm and is 0.4 mm thick. The foil was folded in half, and then in half again, and so on. The thickness (in mm) of the shape formed when the foil was folded n times was modelled using a geometric sequence.

- a** Calculate the thickness, in mm, of the shape when the foil is folded four times.
b Calculate the volume, in cm^3 , of the shape when the foil is folded four times. You may assume after all folds the shape formed was a cuboid.
c Calculate the thickness of the shape if the foil were folded 40 times. Compare your answer with the distance from the Earth to the Moon (approximately 384 000 km).

- 10** At the end of the first month of opening, the electricity bill for a small business was £450. The amount due increased by 5% each month. The bills were analysed and it was correctly stated that, to the nearest £10, the average electricity bill per month for the first year was £600

Use a suitable model to determine which average (mean or median, or both) was used in this statement. Justify your answer.

Challenge

- 11** At 1 pm on a particular day, three people were told a rumour. At 2 pm that day, these three people had told three other people the rumour, who, at 3 pm that day, had each told three other people the rumour, and so on. Nobody was told the rumour more than once.

Given that the current estimate for the world's population is 7.4×10^9 , how many hours will it take for the world's population to have heard this rumour?