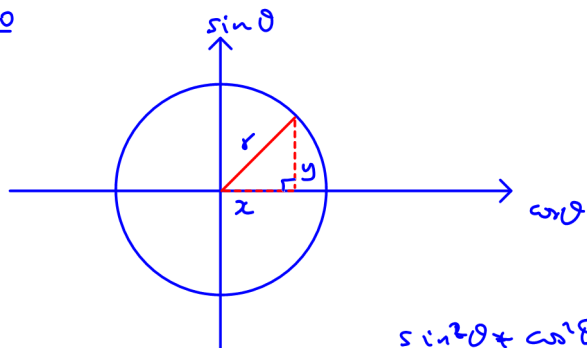


24.10



$$x^2 + y^2 = r^2$$

$$r = 1$$

$$x = \cos \theta$$

$$y = \sin \theta$$

$$\sin^2 \theta + \cos^2 \theta = 1 \rightarrow \sin^2 \theta = 1 - \cos^2 \theta$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

e.g. $\theta = 30^\circ$

$$(\sin 30) ^2 + (\cos 30) ^2$$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta}$$

$$= \frac{1}{4} + \frac{3}{4} = \frac{4}{4} = 1$$

Example 6

Simplify the following expressions:

a $\sin^2 3\theta + \cos^2 3\theta$

b $5 - 5 \sin^2 \theta$

c $\frac{\sin 2\theta}{\sqrt{1 - \sin^2 2\theta}}$

a) 1

b) $5 - 5(1 - \cos^2 \theta)$

$$= 5 - 5 + 5 \cos^2 \theta$$

$$= 5 \cos^2 \theta$$

c) $\frac{\sin 2\theta}{\sqrt{\cos^2 2\theta}} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$

Example 7

Prove that $\frac{\cos^4 \theta - \sin^4 \theta}{\cos^2 \theta} \equiv 1 - \tan^2 \theta$

LHS: $\frac{(\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta)}{\cos^2 \theta} = \frac{\cos^2 \theta}{\cos^2 \theta} - \frac{\sin^2 \theta}{\cos^2 \theta}$

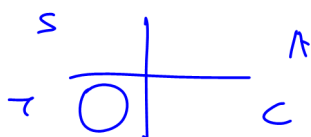
$$= 1 - \tan^2 \theta \equiv \text{RHS}$$

Example 8

a Given that $\cos \theta = -\frac{3}{5}$ and that θ is reflex, find the value of $\sin \theta$.

b Given that $\sin \alpha = \frac{2}{5}$ and that α is obtuse, find the exact value of $\cos \alpha$.

a) reflex (3rd or 4th) since $\cos \theta$ is -ve (3rd quadrant) only $\tan \theta$ positive

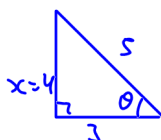


$$\sin \theta = -\sqrt{1 - \cos^2 \theta}$$

$$= -\sqrt{1 - \left(-\frac{3}{5}\right)^2}$$

$$= -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

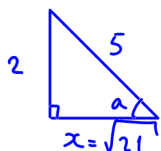
or



$$\sin \theta = -\frac{4}{5}$$

b Given that $\sin \alpha = \frac{2}{5}$ and that α is obtuse, find the exact value of $\cos \alpha$.

2nd quad. only sin +ve



$$\cos \alpha = -\frac{\sqrt{21}}{5}$$

Example 9

Given that $p = 3 \cos \theta$, and that $q = 2 \sin \theta$, show that $4p^2 + 9q^2 = 36$.

$$\begin{aligned} &= 4(3 \cos \theta)^2 + 9(2 \sin \theta)^2 \\ &= 4 \times 9 \cos^2 \theta + 9 \times 4 \sin^2 \theta \\ &= 36(\cos^2 \theta + \sin^2 \theta) \\ &= 36 \times 1 \end{aligned}$$

5 Using the identities $\sin^2 A + \cos^2 A \equiv 1$ and/or $\tan A = \frac{\sin A}{\cos A}$ ($\cos A \neq 0$), prove that:

a $(\sin \theta + \cos \theta)^2 \equiv 1 + 2 \sin \theta \cos \theta$

b $\frac{1}{\cos \theta} - \cos \theta \equiv \sin \theta \tan \theta$

c $\tan x + \frac{1}{\tan x} \equiv \frac{1}{\sin x \cos x}$

d $\cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$

e $(2 \sin \theta - \cos \theta)^2 + (\sin \theta + 2 \cos \theta)^2 \equiv 5$

f $2 - (\sin \theta - \cos \theta)^2 \equiv (\sin \theta + \cos \theta)^2$

g $\sin^2 x \cos^2 y - \cos^2 x \sin^2 y \equiv \sin^2 x - \sin^2 y$

c) LHS $\therefore \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} \equiv \text{RHS}$

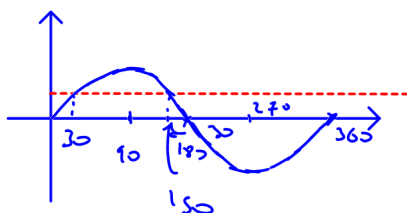
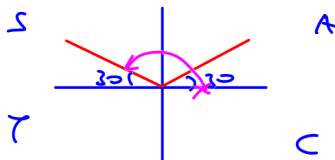
d) LHS $\therefore \sin^2 x (1 - \sin^2 y) - (1 - \sin^2 x) \sin^2 y$
 $\sin^2 x - \cancel{\sin^2 x \sin^2 y} - \sin^2 y + \cancel{\sin^2 x \sin^2 y}$
 $\equiv \text{RHS}$

Example 10

Find the solutions of the equation $\sin \theta = \frac{1}{2}$ in the interval $0 \leq \theta \leq 360^\circ$.

$\sin \theta$ +ve (1st & 2nd)

$\theta = 30^\circ, 150^\circ$

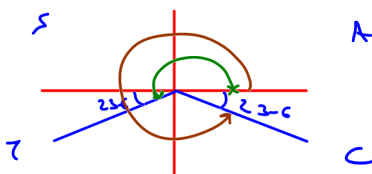


Example 11

Solve, in the interval $0 \leq x \leq 360^\circ$, $5 \sin x = -2$. *-ve (3rd, 4th)*

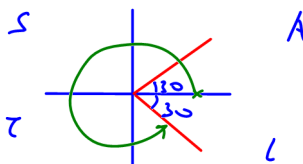
$$x = \sin^{-1}(-2/5)$$

$$= \cancel{-23.6}, 180 + 23.6, 360 - 23.6$$

**Example 12**

Solve, in the interval $0 < x \leq 360^\circ$, $\cos x = \frac{\sqrt{3}}{2}$ *(1st & 4th)*

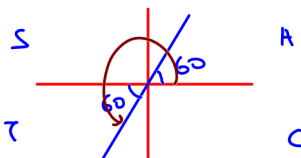
$$x = 30, 330$$

**Example 13**

Find the values of θ in the interval $0 < \theta \leq 360^\circ$ that satisfy the equation $\frac{\sin \theta}{\cos \theta} = \sqrt{3} \frac{\cos \theta}{\cos \theta}$.

$$\tan \theta = \sqrt{3} \quad (\text{1st, 3rd})$$

$$\theta = 60^\circ, 180 + 60$$

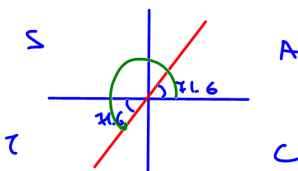


- 9 Find all the values of θ , to 1 decimal place, in the interval $0 < \theta < 360^\circ$ for which $\tan^2 \theta = 9$. **(5 marks)**

$$\tan \theta = \pm 3$$

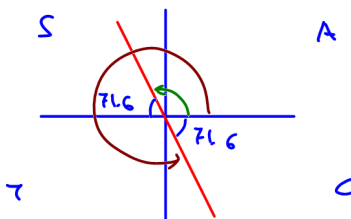
$$\tan \theta = 3 \quad (\text{1st & 3rd})$$

$$\theta = 71.6^\circ, 251.6$$



$$\tan \theta = -3 \quad (\text{2nd & 4th})$$

$$\theta = \cancel{-71.6}, 180 - 71.6, 360 - 71.6$$



10 a Show that $4 \sin^2 x - 3 \cos^2 x = 2$ can be written as $7 \sin^2 x = 5$. (2 marks)

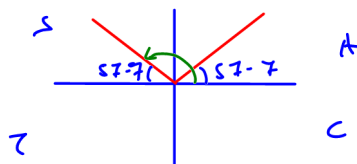
b Hence solve, for $0 \leq x \leq 360^\circ$, the equation $4 \sin^2 x - 3 \cos^2 x = 2$.
Give your answers to 1 decimal place. (7 marks)

$$4 \sin^2 x - 3(1 - \sin^2 x) = 2$$

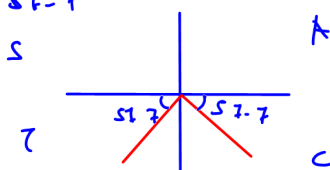
$$4 \sin^2 x - 3 + 3 \sin^2 x = 2 \Rightarrow 7 \sin^2 x = 5$$

$$b) \sin x = \pm \sqrt{\frac{5}{7}}$$

$$\sin x = \sqrt{\frac{5}{7}} \quad x = 51.7^\circ, 180 - 51.7$$



$$\sin x = -\sqrt{\frac{5}{7}} \quad x = -51.7, 180 + 51.7, 360 - 51.7$$



Example 14

a Solve the equation $\cos 3\theta = 0.766$, in the interval $0 \leq \theta \leq 360^\circ$.

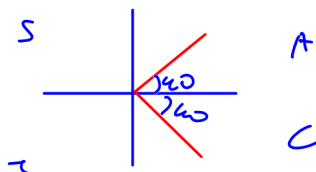
b Solve the equation $2 \sin 2\theta = \cos 2\theta$, in the interval $0 \leq \theta \leq 360^\circ$.

a) Interval $0 \leq 3\theta \leq 1080$

$$3\theta = 40, 320 \text{ (1st)}$$

$$, 40, 680 \text{ 2nd}$$

$$760, 1040$$



$$\theta = \frac{40}{3}, \frac{320}{3}, \frac{400}{3}, \frac{680}{3}, \frac{760}{3}, \frac{1040}{3}$$

b Solve the equation $2 \sin 2\theta = \cos 2\theta$, in the interval $0 \leq \theta \leq 360^\circ$.

$$0 \leq 2\theta \leq 720$$

$$\tan 2\theta = \frac{1}{2}$$

$$\frac{2\theta}{2} = \frac{26.6}{2}, \frac{180 + 26.6}{2}, \frac{360 + 26.6}{2}, \frac{360 + 180 + 26.6}{2}$$

Example 16

Solve for θ , in the interval $0 \leq \theta \leq 360^\circ$, the equations

a $2 \cos^2 \theta - \cos \theta - 1 = 0$

b $\sin^2(\theta - 30^\circ) = \frac{1}{2}$

$-30 \leq \theta - 30 \leq 330$

$(\cos \theta - 1)(2 \cos \theta + 1) = 0$

b) $\sin \theta = \pm \frac{1}{\sqrt{2}}$

$\cos \theta = 1, \theta = 0, 360$

$\theta - 30 = \frac{1}{\sqrt{2}}, \theta - 30 = 45, 135 \Rightarrow \theta = 75, 165$

$\cos \theta = -1/2, \theta = 120, 240$

$\sin(\theta - 30) = -\frac{1}{\sqrt{2}}, \theta - 30 = -45, 135, 315 \Rightarrow \theta = 15, 165, 345$

$\theta = 75, 155, 165, 345$

Example 17

Find the values of x , in the interval $-180^\circ \leq x \leq 180^\circ$, satisfying the equation

$2 \cos^2 x + 9 \sin x = 3 \sin^2 x$

$2(1 - \sin^2 x) + 9 \sin x = 3 \sin^2 x$

$2 - 2 \sin^2 x + 9 \sin x = 3 \sin^2 x \Rightarrow 5 \sin^2 x - 9 \sin x - 2 = 0$

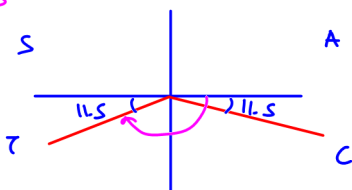
$(\sin x - 2)(5 \sin x + 1) = 0$

$\sin x = 2$ not possible

$\sin x = -1/5$

$-180^\circ \leq x \leq 180^\circ$

$x = -11.5, -180 \leq x \leq 11.5$



11 The vectors $2\mathbf{a} + k\mathbf{b}$ and $5\mathbf{a} + 3\mathbf{b}$ are parallel. Find the value of k .

$\begin{pmatrix} 2a + kb \\ x \end{pmatrix} \parallel \begin{pmatrix} 5a + 3b \\ y \end{pmatrix}$

$2a + kb = \lambda(5a + 3b)$

or $2a + kb = 5\lambda a + 3\lambda b$

$\frac{k}{2} = \frac{3}{5} \Rightarrow k = \frac{6}{5}$

$2 = 5\lambda \Rightarrow \lambda = \frac{2}{5}$
 $k = 3\lambda \Rightarrow k = \frac{6}{5}$

Example

7

$$\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}, \mathbf{b} = 2\mathbf{i} + 7\mathbf{j}$$

$$\text{Find } \mathbf{a} + \frac{1}{2}\mathbf{a} \quad \mathbf{b} + \mathbf{a} + \mathbf{b} \quad \mathbf{c} \ 3\mathbf{a} - 2\mathbf{b} =$$

$$\begin{aligned} \mathbf{c) \ 3a - 2b} &= 3(3\mathbf{i} - 4\mathbf{j}) - 2(2\mathbf{i} + 7\mathbf{j}) \\ &= 9\mathbf{i} - 12\mathbf{j} - 4\mathbf{i} - 14\mathbf{j} \\ &= 5\mathbf{i} - 26\mathbf{j} \end{aligned}$$

4 Given that $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$, find:

a λ if $\mathbf{a} + \lambda\mathbf{b}$ is parallel to the vector $\mathbf{i}$

b μ if $\mu\mathbf{a} + \mathbf{b}$ is parallel to the vector $\mathbf{j}$

$$\begin{aligned} \mathbf{a) \ a + \lambda b} &= (2 + 3\lambda)\mathbf{i} + (5 - \lambda)\mathbf{j} \\ \text{parallel to vector } \mathbf{j} &\longrightarrow \\ \text{vector } \mathbf{j} \text{ is zero} &\longrightarrow \mathbf{i} \\ 5 - \lambda &= 0 \\ \lambda &= 5 \end{aligned}$$

$$\begin{aligned} \mathbf{b) \ \mu a + b} &= (2\mu + 3)\mathbf{i} + (5\mu - 1)\mathbf{j} \\ \text{vector is } \mathbf{i} \text{ zero} &\longrightarrow \\ 2\mu + 3 &= 0 \quad \mu = -3/2 \end{aligned}$$

5 Given that $\mathbf{c} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{d} = \mathbf{i} - 2\mathbf{j}$, find:

a λ if $\mathbf{c} + \lambda\mathbf{d}$ is parallel to $\mathbf{i} + \mathbf{j}$

b μ if $\mu\mathbf{c} + \mathbf{d}$ is parallel to $\mathbf{i} + 3\mathbf{j}$

c s if $\mathbf{c} - s\mathbf{d}$ is parallel to $2\mathbf{i} + \mathbf{j}$

d t if $\mathbf{d} - t\mathbf{c}$ is parallel to $-2\mathbf{i} + 3\mathbf{j}$

$$\begin{aligned} \mathbf{c) \ c - sd} &\parallel 2\mathbf{i} + \mathbf{j} \\ \mathbf{c - sd} &= (3 - s)\mathbf{i} + (4 + 2s)\mathbf{j} \\ \frac{4 + 2s}{3 - s} &= \frac{1}{2} \\ 8 + 4s &= -2 \\ s &= -\frac{5}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{d) \ d - tc} &= (1 - 3t)\mathbf{i} + (-2 - 4t)\mathbf{j} \\ \frac{-2 - 4t}{1 - 3t} &= \frac{3}{-2} \\ 4 + 8t &= 3 - 9t \\ 17t &= -1 \quad t = -\frac{1}{17} \end{aligned}$$

$$\mathbf{8 \ a} = \begin{pmatrix} j \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 10 \\ k \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Given that $\mathbf{b} - 2\mathbf{a} = \mathbf{c}$, find the values of j and k .

(2 marks)

$$\mathbf{b - 2a} = \begin{pmatrix} 10 \\ k \end{pmatrix} - 2 \begin{pmatrix} j \\ 3 \end{pmatrix} = \begin{pmatrix} 10 - 2j \\ k - 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\begin{aligned} 10 - 2j &= 2 \Rightarrow j = 4 \\ k - 6 &= 5 \Rightarrow k = 11 \end{aligned}$$

10 The resultant of the vectors $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = p\mathbf{i} - 2p\mathbf{j}$ is parallel to the vector $\mathbf{c} = 2\mathbf{i} - 3\mathbf{j}$. Find:

a the value of p

(4 marks)

b the resultant of vectors $\mathbf{a}$ and $\mathbf{b}$.

(1 mark)

$$\text{Result: } \mathbf{a} + \mathbf{b} = (3+p)\mathbf{i} + (-2-2p)\mathbf{j} \quad // \quad \mathbf{c} = 2\mathbf{i} - 3\mathbf{j}$$

$$\frac{-2-2p}{3+p} = \frac{-3}{2} \Rightarrow -4-4p = -9-3p$$

$$p = 5$$

$$\text{b) } \mathbf{a} + \mathbf{b} = 8\mathbf{i} - 12\mathbf{j}$$