

## 8 The binomial expansion

### 8.1 Pascal's triangle

- 1 Use Pascal's triangle to write down the expansion of:

a  $(3 - x)^5$                       b  $(2x + y)^4$   
c  $(3x - 2y)^5$

**Hint** The coefficients in the expansion of  $(a + b)^n$  are in the  $(n + 1)$ th row of Pascal's triangle.

- 2 Find the coefficient of  $x^3$  in the expansion of:

a  $(5 + 2x)^4$                       b  $(2 - 3x)^5$   
c  $\left(3 - \frac{1}{2}x\right)^4$

**Hint** The term in  $x^n$  is the  $(n + 1)$ th term in the expansion in ascending powers of  $x$ . When substituting a term with two or more characters, such as  $2x$ , into the expansion of  $(a + b)^n$  use brackets to make sure you don't make a mistake.

- 3 Fully expand the expression  $(1 - 2x)(1 + 3x)^3$ .

**Hint** Expand  $(1 + 3x)^3$ , then multiply each term by 1 and by  $-2x$ .

- E** 4 a Find all the terms of the expansion of  $(2 + 3x)^4$ . Give each term in its simplest form. (2 marks)  
b Write down the expansion of  $(2 - 3x)^4$  in ascending powers of  $x$ , giving each term in its simplest form. (1 mark)
- E/P** 5 The coefficient of  $x^2$  in the expansion of  $(3 + kx)^4$  is 1350. Find two possible values of the constant,  $k$ . (3 marks)
- E/P** 6 a Expand  $(4 + k)^3$ . (2 marks)  
b Hence, or otherwise, write down the expansion of  $(4 + x^2 - x)^3$  in ascending powers of  $x$ . (3 marks)
- E/P** 7 The coefficient of  $x^2$  in the expansion of  $(5 + x)(2 - kx)^3$  is 18. Find two possible values of the constant,  $k$ . (3 marks)

### 8.2 Factorial notation

- 1 Without using a calculator, work out:

a  $0!5!$       b  $\frac{12!}{9!}$       c  $\frac{20!}{18!}$

**Hint** By definition,  $0! = 1$ . For parts **b** and **c**, you can simplify by cancelling. For example,  $12!$  can be written as  $12 \times 11 \times 10 \times 9!$

- 2 Use a calculator to work out:

a  $\binom{16}{9}$                       b  ${}^{12}C_5$   
c  $\binom{22}{10}$                       d  ${}^{18}C_{11}$

**Hint**  ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ ,  $n, r \in \mathbb{N}$ .

You can use the  ${}^nC_r$  and  $!$  functions on your calculator.

- 3 Write each value,  $p$ ,  $q$ ,  $r$  and  $s$ , from Pascal's triangle using  ${}^nC_r$  notation.

|   |   |   |    |     |     |   |     |   |   |
|---|---|---|----|-----|-----|---|-----|---|---|
|   |   |   | 1  |     |     |   |     |   |   |
|   |   |   | 1  |     | 1   |   |     |   |   |
|   |   | 1 |    | 2   |     | 1 |     |   |   |
|   | 1 |   | 3  |     | 3   |   | 1   |   |   |
|   | 1 | 4 |    | $p$ |     | 4 |     | 1 |   |
| 1 |   | 5 | 10 |     | $q$ |   | 5   |   | 1 |
| 1 | 6 |   | 15 |     | $r$ |   | $s$ |   | 6 |
|   |   |   |    |     |     |   |     |   | 1 |

- 4 Write the 8th number in the 15th row of Pascal's triangle in  ${}^nC_r$  form.

**Hint** The  $r$ th entry in the  $n$ th row of Pascal's triangle is given by  ${}^{n-1}C_{r-1}$  or  $\binom{n-1}{r-1}$ .

- 5 The probability of rolling  $n$  threes when a fair dice is rolled 10 times is given by

$$\binom{10}{n} \left(\frac{1}{6}\right)^n \left(\frac{5}{6}\right)^{10-n}, 0 \leq n \leq 10$$

Find the probability of rolling:

- a no threes                      b 5 threes

**E** 6 Given that  $\binom{25}{11} = \frac{25!}{11!p!}$ , write down the value of  $p$ . (1 mark)

**E** 7 Given that  $\binom{30}{q} = \frac{30!}{q!8!}$ , write down the value of  $q$ . (1 mark)

**E/P** 8 a Show that  ${}^{k+2}C_k = \frac{(k+2)(k+1)}{2}$  (3 marks)

b Given that  ${}^{k+2}C_k = 36$ , find the value of  $k$ . (3 marks)

c Explain why there is only one solution for  $k$  in part b. (1 mark)

### 8.3 The binomial expansion

- 1 Use the binomial theorem to find the expansion of:

a  $(1+x)^5$                       b  $(2+x)^6$                       c  $(10+x)^4$

- 2 Use the binomial theorem to find the expansion of:

a  $(3-2x)^4$                       b  $\left(4+\frac{1}{2}x\right)^5$

c  $(x-2)^6$

**Hint** The binomial expansion is:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n \quad (n \in \mathbb{N})$$

$$\text{where } {}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}, \quad (n, r \in \mathbb{N})$$

- 3 Use the binomial theorem to find the first four terms, in ascending powers of  $x$ , in the expansion of:

a  $(1+x)^{10}$                       b  $(1+x)^{15}$                       c  $(2+x)^9$

**Hint** You need to expand each expression up to the term in  $x^3$ .

- 4 Use the binomial theorem to find the first four terms, in ascending powers of  $x$ , in the expansion of:

a  $\left(1 - \frac{x}{2}\right)^{10}$

b  $\left(2 - \frac{1}{3}x\right)^9$

c  $(3 + 2x)^8$

**Hint** Use brackets when you substitute terms into the binomial expansion formula, and include any negative signs.

For example, the 4th term of  $\left(2y - \frac{x}{2}\right)^5$  is

$$\binom{5}{3}(2y)^2\left(-\frac{x}{2}\right)^3 = 10(4y^2)\left(-\frac{x^3}{8}\right) = -5y^2x^3$$

- (E)** 5 Find the first four terms, in ascending powers of  $x$ , of the binomial expansion of  $(2 - 3x)^5$ , giving each term in its simplest form. **(4 marks)**
- (E)** 6 Given that  $(3 - x)^{12} \equiv A + Bx + Cx^2 + \dots$ , find the values of the integers  $A$ ,  $B$  and  $C$ . **(4 marks)**
- (E)** 7 Find the first four terms, in ascending powers of  $x$ , of the binomial expansion of  $\left(1 - \frac{2}{3}x\right)^9$ . **(4 marks)**
- (E/P)** 8 Find the binomial expansion of  $\left(x - \frac{2}{x}\right)^5$ , giving each term in its simplest form. **(4 marks)**

## 8.4 Solving binomial problems

- 1 Find the coefficient of  $x^3$  in the expansion of:

a  $(1 + x)^{14}$

b  $(2 + x)^{10}$

c  $(5 - x)^7$

- 2 Find the coefficient of  $x^4$  in the expansion of:

a  $(3 - 2x)^9$

b  $(5 + 3x)^7$

c  $\left(\frac{1}{2} + 8x\right)^{12}$

**Hint** The **general term** in the expansion of  $(a + b)^n$  is  $\binom{n}{r}a^{n-r}b^r$

- 3 Find the coefficient of  $x^3$  in the expansion of  $(3 - x)(2 + 3x)^6$ .

**Hint** First expand  $(2 + 3x)^6$  up to the term in  $x^3$  then multiply by  $(3 - x)$ .

There are two ways of making a term in  $x^3$ :  
(constant term  $\times x^3$  term) and ( $x$  term  $\times x^2$  term)

- (E)** 4 Find the coefficient of  $x^3$  in the expansion of  $\left(1 + \frac{2}{3}x\right)^6$  **(3 marks)**
- (E/P)** 5 The coefficient of  $x^2$  in the expansion of  $(1 + 2x)^n$  is 60. Given that  $n > 0$ , find the value of  $n$ . **(3 marks)**
- (P)** 6 The coefficient of  $x^2$  in the expansion of  $(3 + ax)^5$  is 4320. Find two possible values of the constant,  $a$ .
- (P)** 7 The coefficient of  $x^3$  in the expansion of  $(2 - bx)^6$  is  $-20$ . Find the value of the constant,  $b$ .

- E/P 8** a Find the first four terms, in ascending powers of  $x$ , of the binomial expansion of  $(2 - 3x)^8$ , giving each term in its simplest form. (4 marks)
- b Hence, or otherwise, find the coefficient of  $x^3$  in the expansion of  $(1 + \frac{x}{2})(2 - 3x)^8$ . (3 marks)
- E/P 9** a Find the first three terms, in ascending powers of  $x$ , of the binomial expansion of  $(3 + px)^6$ , where  $p$  is a non-zero constant. Give each term in its simplest form. (4 marks)
- b Given that, in this expansion, the coefficient of  $x^2$  is 5 times the coefficient of  $x$ , find the value of  $p$ . (2 marks)

### 8.5 Binomial estimation

- 1** a Find the first four terms, in ascending powers of  $x$ , of the binomial expansion of  $(1 - \frac{x}{5})^9$ . Give each term in its simplest form.
- b Use your expansion to estimate the value of  $0.985^9$ , giving your answer to 4 decimal places.
- Hint** For part **b**, work out the value of  $x$  to substitute into the expansion by letting  $1 - \frac{x}{5} = 0.985$ .
- 2** a Find the first three terms, in ascending powers of  $x$ , of the binomial expansion of  $(1 - 2x)^6$ . Give each term in its simplest form.
- b If  $x$  is so small that  $x^3$  and higher powers can be ignored, show that  $(3 + x)(1 - 2x)^6 \approx 3 - 35x + 168x^2$ .
- Hint** Multiply your answer to part **a** by  $(3 + x)$  and collect like terms.
- 3** a Write down the first four terms, in ascending powers of  $x$ , of the binomial expansion of  $(3 + \frac{x}{4})^8$ .
- b By substituting an appropriate value for  $x$ , find an approximate value for  $3.05^8$ .
- Hint** For part **b**, work out the value of  $x$  to substitute into the expansion by letting  $\frac{x}{4} = 0.05$ .
- E 4** a Find the first four terms, in ascending powers of  $x$ , of the binomial expansion of  $(1 + \frac{x}{4})^9$ . Give each term in its simplest form. (4 marks)
- b Use your expansion to estimate the value of  $1.015^9$ , giving your answer to 3 decimal places. (3 marks)
- E/P 5** a Find the first four terms, in ascending powers of  $x$ , of the binomial expansion of  $(1 - 3x)^6$ . Give each term in its simplest form. (4 marks)
- b If  $x$  is small, so that  $x^3$  and higher powers can be ignored, show that  $(1 + \frac{x}{2})(1 - 3x)^6 \approx 1 - \frac{35x}{2} + 126x^2$  (2 marks)

- E/P 6** a Find the first three terms, in ascending powers of  $x$ , of the binomial expansion of  $\left(2 - \frac{x}{2}\right)^7$ , giving each term in its simplest form. **(4 marks)**
- b Explain how you would use your expansion to give an estimate for the value of  $1.985^7$ . **(1 mark)**
- E/P 7** a Find the first four terms, in ascending powers of  $x$ , of the binomial expansion of  $\left(3 - \frac{x}{5}\right)^{10}$ . Give each term in its simplest form. **(4 marks)**
- b Use your answer to part a to estimate the value of  $2.995^{10}$ , giving your answer to 6 decimal places. **(3 marks)**
- c Use your calculator to evaluate  $2.995^{10}$  and calculate the percentage error in your answer to part b. **(2 marks)**

### Problem solving Set A

#### Bronze

In the binomial expansion of  $(3 - kx)^6$ , where  $k$  is a constant, the coefficient of  $x^3$  is  $-4320$ . Calculate:

- a the value of  $k$  **(4 marks)**
- b the value of the coefficient of  $x^4$  in the expansion. **(2 marks)**

#### Silver

- a Write down the first three terms, in ascending powers of  $x$ , of the binomial expansion of  $(2 - px)^{11}$ , where  $p$  is a non-zero constant, giving each term in its simplest form. **(4 marks)**
- b Given that, in the expansion of  $(2 - px)^{11}$ , the coefficient of  $x$  is  $q$  and the coefficient of  $x^2$  is  $10q$ , find the value of  $p$  and the value of  $q$ . **(3 marks)**

#### Gold

In the binomial expansion of  $\left(3 + \frac{x}{k}\right)^8$ , the coefficient of  $x^2$  is 3 times the coefficient of  $x^3$ . Find the value of the constant,  $k$ . **(7 marks)**

## Problem solving Set B

## Bronze

$f(x) = (2 + kx)(1 - 5x)^4$ , where  $k$  is a constant.

The expansion, in ascending powers of  $x$ , of  $f(x)$  up to and including the term in  $x^2$  is

$A - 37x + Bx^2$ , where  $A$  and  $B$  are constants.

- a Find the first three terms, in ascending powers of  $x$ , of the binomial expansion of  $(1 - 5x)^4$ , giving each term in its simplest form. (4 marks)
- b Write down the value of  $A$ . (1 mark)
- c Find the value of  $k$ . (2 marks)
- d Hence find the value of  $B$ . (2 marks)

## Silver

$f(x) = (1 + kx)(2 - 5x)^4$ , where  $k$  is a constant.

The expansion, in ascending powers of  $x$ , of  $f(x)$  up to and including the term in  $x^2$  is

$A - 64x + Bx^2$ , where  $A$  and  $B$  are constants.

Find the values of  $A$ ,  $B$  and  $k$ . (7 marks)

## Gold

$f(x) = (3 + px)(1 + qx)^5$ , where  $p$  and  $q$  are positive constants.

The expansion, in ascending powers of  $x$ , of  $f(x)$  up to and including the term in  $x^3$  is

$3 + 17x + \frac{70}{3}x^2 + kx^3$  where  $k$  is a constant.

- a Find the value of  $p$  and the value of  $q$ . (7 marks)
- b Hence find the value of  $k$ . (1 mark)

**Now try this** → Exam question bank Q9, Q20, Q29, Q40, Q51, Q61