

7.1 Algebraic fractions

1 Simplify these fractions:

a $\frac{6x^4 + 8x^3 - 4x^2}{2x}$

b $\frac{12x - 24x^2 + 9x^3}{3x}$

c $\frac{8x^6 - 4x^5 + 2x^4}{-2x^2}$

Hint Divide each term in the numerator by the denominator.

2 Write each of these fractions in their simplest form:

a $\frac{x^2 - 9}{x - 3}$

b $\frac{x^2 - 3x + 2}{x - 2}$

c $\frac{2x^2 + 5x - 3}{x + 3}$

Hint To simplify an algebraic fraction, factorise the numerator and denominator where possible and cancel common factors.

3 Write each of these fractions in their simplest form:

a $\frac{2x^2 - 3x - 2}{x^2 + x - 6}$

b $\frac{3x^2 + x - 2}{2x^2 + 5x + 3}$

c $\frac{6x^2 + 11x - 10}{2x^2 - 3x - 20}$

E/P 4 Show that $\frac{15x^2 - 16x - 15}{3x^2 - 5x}$ can be written in the form $A + \frac{B}{x}$, where A and B are integers to be found. (3 marks)

E/P 5 $\frac{x^2 - 2x + k}{x^2 + 2x - 3} = \frac{x - 5}{x - 1}$ where k is a constant. Work out the value of k . (3 marks)

E/P 6 $\frac{4x^2 + 8x - 5}{6x^2 - 7x + k} = \frac{2x + 5}{3x - 2}$ where k is a constant. Work out the value of k . (4 marks)

E/P 7 $\frac{6x^3 - 2x^2 - 4x}{2x^2 - 7x + 5} = \frac{ax(bx + c)}{px + q}$, where a, b, c, p and q are constants.

 Work out the values of a, b, c, p and q . (5 marks)

7.2 Dividing polynomials

 1 Write each polynomial in the form $(x \pm p)(ax^2 + bx + c)$ by dividing:

a $2x^3 - 5x^2 + 8x - 5$ by $(x - 1)$

b $3x^3 + 8x^2 + 3x - 2$ by $(x + 2)$

c $2x^3 + x^2 - 17x - 12$ by $(x - 3)$

d $4x^3 + 13x^2 - 11x + 4$ by $(x + 4)$

Hint You can use long division to divide a polynomial by $(x \pm p)$, where p is a constant.

For example:

$$\begin{array}{r} 2x^2 \\ x-1 \overline{) 2x^3 - 5x^2 + 8x - 5} \\ \underline{2x^3 - 2x^2} \\ -3x^2 + 8x - 5 \end{array}$$

and so on

2 Divide:

a $3x^4 + 8x^3 - x^2 - 13x - 6$ by $(x + 2)$

b $4x^4 - 8x^3 + x^2 - x - 2$ by $(2x + 1)$

c $9x^4 - 3x^3 - 17x^2 + 13x - 2$ by $(3x - 2)$

d $4x^4 - 12x^3 - 5x^2 + 15x + 9$ by $(2x - 3)$

Hint The answer you obtain following the division is called the **quotient**.

3 Divide:

- a $2x^3 + 6x^2 - 4$ by $(x + 1)$
- b $3x^3 + 7x^2 + 18$ by $(x + 3)$
- c $4x^3 - 11x - 10$ by $(x - 2)$
- d $2x^3 + 7x^2 + 75$ by $(x + 5)$

Hint Write the polynomial in part a as $2x^3 + 6x^2 + 0x - 4$ before dividing.

4 Find the remainder when:

- a $x^3 + 3x^2 + 5x - 8$ is divided by $(x + 4)$
- b $2x^3 - 5x^2 + 12x - 20$ is divided by $(x - 3)$
- c $3x^3 + 2x^2 - 40x + 45$ is divided by $(x + 5)$

Hint If there is a remainder, then the linear expression $(x \pm p)$ is not a factor. The polynomial can be written as $(x \pm p)(ax^2 + bx + c) + r$ where r is the remainder.

E 5 Find the remainder when $-15x^3 + 26x^2 - 13x + 5$ is divided by $(5x - 2)$. **(2 marks)**

E/P 6 $f(x) = 6x^3 - 13x^2 - 13x + 30$

- a Find the remainder when $f(x)$ is divided by $(x + 3)$. **(2 marks)**
- b Given that $(x - 2)$ is a factor of $f(x)$, factorise $f(x)$ completely. **(4 marks)**

E/P 7 $f(x) = 2x^3 + 3x^2 - 4x + k$ where k is a constant.

Given that $(x + 3)$ is a factor of $f(x)$:

- a find the value of k **(2 marks)**
- b express $f(x)$ in the form $(x + 3)(ax^2 + bx + c)$ where a , b and c are constants **(2 marks)**
- c show that $f(x) = 0$ has exactly one real solution. **(2 marks)**

E/P 8 $f(x) = 3x^3 + 10x^2 - 8x - 5$

- a Find the remainder, r , when $f(x)$ is divided by $(x - 2)$. **(2 marks)**
- b Express $f(x)$ in the form $(x - 2)(ax^2 + bx + c) + r$ where a , b , c and r are constants. **(4 marks)**

E/P 9 $f(x) = 10x^3 - 29x^2 + 4x + 15$

- a Given that $(x - 1)$ is a factor of $f(x)$, express $f(x)$ in the form $(x - 1)(ax^2 + bx + c)$, where a , b and c are constants. **(2 marks)**
- b Hence factorise $f(x)$ completely. **(4 marks)**
- c Write down all the solutions to the equation $f(x) = 0$. **(2 marks)**

7.3 The factor theorem

1 Use the factor theorem to show that:

- a $(x + 1)$ is a factor of $2x^3 + 7x^2 - 5$
- b $(x + 2)$ is a factor of $x^3 + 4x^2 + 3x - 2$
- c $(x - 3)$ is a factor of $2x^3 - 3x^2 - 7x - 6$
- d $(x - 4)$ is a factor of $x^4 - 3x^3 - 15x - 4$

Hint The **factor theorem** states that if $f(x)$ is a polynomial, then:

- if $f(p) = 0$ then $(x - p)$ is a factor of $f(x)$
- if $(x - p)$ is a factor of $f(x)$ then $f(p) = 0$

- 2 Use the factor theorem to show that the linear expression is a factor of the polynomial $f(x)$ and factorise $f(x)$ completely:

- a $(x - 2)$, $2x^3 + x^2 - 13x + 6$
- b $(x + 3)$, $2x^3 + 17x^2 + 38x + 15$
- c $(x - 1)$, $6x^3 - x^2 - 11x + 6$
- d $(x + 4)$, $15x^3 + 61x^2 - 2x - 24$

- 3 Fully factorise each expression:

- a $x^3 + 2x^2 - 21x + 18$
- b $2x^3 + 13x^2 + 13x - 10$
- c $3x^3 + 2x^2 - 41x - 60$

- 4 For each of the following polynomials,

- i fully factorise each polynomial $f(x)$.
- ii Hence sketch the graph of $y = f(x)$.

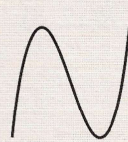
- a $2x^3 - 11x^2 + 5x + 18$
- b $2x^3 - 3x^2 - 39x + 20$
- c $6x^3 + 37x^2 + 50x - 21$

Hint When you have used the factor theorem to show the linear expression is a factor, you can use long division to find the quadratic factor. Factorise the quadratic factor to write the polynomial as a product of three linear factors.

Hint Try values of p in each expression for $f(x)$, e.g. $p = -1, 1, 2, 3, \dots$ until you find $f(p) = 0$. Then use the factor theorem to deduce that $(x - p)$ is a factor of $f(x)$.

Hint To sketch the graph, you need to identify the points where the curve crosses the axes. Set $x = 0$ to find the y -intercept and $y = 0$ to find the x -intercepts.

The general shapes of cubic graphs are:



if the coefficient of x^3 is positive



if the coefficient of x^3 is negative

← Section 4.1

- E/P** 5 $f(x) = 6x^3 - 17x^2 - 15x + 36$
Given that $(x - 3)$ is a factor of $f(x)$, find all the solutions to $f(x) = 0$. (5 marks)

- E/P** 6 $f(x) = 9x^3 + 24x^2 - 44x + 16$
- a Use the factor theorem to show that $(x + 4)$ is a factor of $f(x)$. (2 marks)
 - b Hence show that $f(x)$ can be written in the form $f(x) = (x + 4)(px + q)^2$, where p and q are integers to be found. (4 marks)

- E** 7 $f(x) = 2x^3 - 3x^2 - 5x + 6$. Factorise $f(x)$ completely. (5 marks)

- E/P** 8 $g(x) = x^3 + 2x^2 - 19x + k$
Given that $(x + 1)$ is a factor of $g(x)$,
- a show that $k = -20$ (2 marks)
 - b express $g(x)$ as a product of three linear factors. (3 marks)
 - c Sketch the curve with equation $y = x^3 + 2x^2 - 19x - 20$, indicating the values where the curve crosses the x -axis and the y -axis. (4 marks)

E/P 9 $p(x) = 25x^3 + 55x^2 - 56x + 12$

- a Use the factor theorem to show that $(x + 3)$ is a factor of $p(x)$. (2 marks)
- b Fully factorise $p(x)$. (3 marks)
- c Hence show that there are exactly two real roots of the equation $p(x) = 0$. (5 marks)

7.4 Mathematical proof

- 1 Prove that $(x - 4)(x + 6)(2x + 3) \equiv 2x^3 + 7x^2 - 42x - 72$

Hint The identity symbol $\equiv$ means 'is equivalent to'. This means the expressions are equal for all values of x . To prove an identity you should:

- start with the expression on one side of the identity
- algebraically manipulate the expression until it matches the other side
- show every step in your algebraic working.

- 2 Prove that the triangle with vertices at $P(10, 14)$, $Q(-6, 2)$ and $R(12, 8)$ is right angled.

Hint In a mathematical proof you must:

- state any information or assumptions you are using
- clearly show every step of your working
- make sure each step follows logically from the previous step
- write a statement of proof at the end of your working.

You could find the lengths of the sides in this triangle and use Pythagoras' theorem, or use gradients of line segments.

← Section 5.4

- 3 Prove that $y = kx$ is a tangent to the curve with equation $y = 4x^2 - 5x + 4$ when $k = 3$ and $k = -13$.

Hint If a line is a tangent to a curve, then there is only one point of intersection. Solve the equations simultaneously and use the discriminant.

← Section 2.5

- 4 Prove that the points $A(-2, 3)$, $B(2, 1)$ and $C(14, -5)$ are collinear.

Hint If points are collinear, they lie on the same straight line.

← Section 5.1

E/P 5 Prove that $x^2 - 6x + 10 > 0$ for all values of x . (3 marks)

E/P 6 Prove that $(x + y)^2 - (x - y)^2 \equiv 4xy$ (3 marks)

- E/P** 7 Prove that the difference between the squares of two consecutive odd numbers is a multiple of 8. (4 marks)
- E/P** 8 Prove that the difference between the cube and the square of an odd number is even. (4 marks)
- E/P** 9 The equation $x^2 + (k - 3)x + (3 - 2k) = 0$ where k is a constant has no real roots. Prove that k satisfies $-3 < k < 1$. (6 marks)

7.5 Methods of proof

- 1 $f(n) = n^2 + n + 17$ for $n \in \mathbb{N}$.
- a Prove that for $5 \leq n \leq 10$, $f(n)$ is prime.
- b Find a counter-example to prove that $f(n)$ is not prime for all positive integers, n .

Hint You can prove a mathematical statement is not true by a **counter-example**. A counter-example is one example that does not work for the statement. One example is sufficient to disprove the statement.

- 2 Prove that the following statement is **not** true.
'The difference of two prime numbers is always even.'
- 3 a Prove that for positive values of x , $(1 + 2x)^2 > 1 + 4x^2$
- b Find a counter-example to prove that the inequality $(1 + 2x)^2 > 1 + 4x^2$ is not true for all values of x .

- 4 A student is trying to prove that $\frac{x(x+2)^2 + x}{x}$, is a positive numbers of all real values of x .

The student writes:

$$\frac{x(x+2)^2 + x}{x} = \frac{x(x+2)^2}{x} + \frac{x}{x} = (x+2)^2 + 1$$

As $(x+2)^2 \geq 0$ for all values of x then $(x+2)^2 + 1 > 0$ for all values of x .

- a Identify the error made in the proof.
- b State the value of x for which the statement is not true.
- E/P** 5 Use a counter-example to show that the following statement is false.
'If you add 2 to a number and square it, the result is always greater than the square of the number.' (2 marks)
- E/P** 6 Prove that $(2n + 1)^2 > 2^{n+1}$ for all positive integers less than 7. (3 marks)
- E/P** 7 Amir claims that $x^2 \geq x$.
Determine whether Amir's claim is always true, sometimes true or never true, justifying your answer. (2 marks)
- E/P** 8 Prove that $(x + 5)^2 \geq 4x + 9$ for all real values of x . (3 marks)
- E/P** 9 Prove that if n is an integer and $2 \leq n \leq 7$, then $n^2 + 2$ is not divisible by 4. (3 marks)

Problem solving Set A

Bronze

$$p(x) = x^3 - 9x^2 + 2x + 48$$

- a Find the remainder, r , when $p(x)$ is divided by $(x - 4)$. (2 marks)
- b Hence write $p(x)$ in the form $(x - 4)(x^2 + ax + b) + r$ where a , b and r are constants. (2 marks)
- c Given that $(x + 2)$ is a factor of $p(x)$, factorise $p(x)$ completely. (4 marks)
- d Prove that if n is an integer and $3 < n < 8$, then $n^3 - 9n^2 + 2n + 48 < 0$ (3 marks)

Silver

$$p(x) = 2x^3 - 15x^2 + 32x + 20$$

Given that $(2x + 1)$ is a factor of $p(x)$,

- a use long division to express $p(x)$ in the form $(2x + 1)(x^2 + ax + b)$, where a and b are constants to be found. (3 marks)
- b Hence show that $p(x) = 0$ has only one real root. (4 marks)
- c Prove that $2x^3 - 15x^2 + 32x + 20 > 0$ for all real positive values of x . (3 marks)

Gold

$$p(x) = 9x^3 - 6x^2 - 20x - 8$$

Given that $p(x) = (x + d)(ax^2 + bx + c)$, where a , b , c and d are integers.

- a find the values of a , b , c and d . (6 marks)
- b Prove that $p(x) = 0$ has exactly two real roots. (3 marks)
- c Sketch the curve with equation $y = 9x^3 - 6x^2 - 20x - 8$, indicating the values where the curve cuts the x -axis and the y -axis. (4 marks)
- d Hence or otherwise, use set notation to write down the solutions to the inequality $9x^3 - 6x^2 < 20x + 8$ (3 marks)

Problem solving Set B

Bronze

$f(x) = x^3 - x^2 + px + q$ where p and q are integers.

Given that $(x + 1)$ is a factor of $f(x)$,

a show that $q - p = 2$. (3 marks)

Given that $(x + 3)$ is also a factor of $f(x)$,

b show that $q - 3p = 36$. (3 marks)

c Hence find the value of p and the corresponding value of q . (2 marks)

d Factorise $f(x)$ completely. (2 marks)

Silver

$f(x) = 2x^3 - x^2 + px + q$ where p and q are integers.

Given that $(x + 2)$ is a factor of $f(x)$,

a show that $q - 2p - 20 = 0$. (3 marks)

Given that $(x - 3)$ is also a factor of $f(x)$,

b find the value of p and the corresponding value of q . (5 marks)

c Factorise $f(x)$ completely. (2 marks)

Gold

$f(x) = x^3 + (p + 4)x^2 + 8x + q$ where p and q are integers.

Given that $(x - 2)$ is a factor of $f(x)$,

a show that $4p + q + 40 = 0$. (3 marks)

Given that $(x + p)$ is also a factor of $f(x)$, and that $p > 0$,

b show that $4p^2 - 8p + q = 0$. (3 marks)

c Hence find the value of p and the corresponding value of q . (5 marks)

d Factorise $f(x)$ completely. (2 marks)

Now try this → Exam question bank Q30, Q34, Q41, Q57, Q69, Q75, Q83