

11 Integration

11.1 Integrating standard functions

1 Integrate the following with respect to x .

- a $e^x + 2 \sin x$ b $4 \cos x - \frac{3}{x^2}$
 c $\frac{5}{x} - 2e^x$ d $\sqrt{x} + \frac{3}{4} \sin x$
 e $11x^2 + \frac{1}{2}e^x - \frac{6}{x}$ f $3 \cos x - \frac{1}{\sqrt{x}}$

Hint Use the following standard integrals to help:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int e^x dx = e^x + c$$

$$\int \frac{1}{x} dx = \ln |x| + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \cos x dx = \sin x + c$$

2 Find the following integrals.

- a $\int \frac{1}{\sin^2 x} dx$ b $\int \frac{\sin x}{\cos^2 x} dx$
 c $\int \frac{\cos x}{\sin^2 x} dx$ d $\int \frac{4}{\cos^2 x} dx$
 e $\int \frac{4}{1 - \cos^2 x} dx$ f $\int \frac{8 \cos x}{\cos^2 x - 1} dx$

Hint Use the following standard integrals to help:

$$\int \sec^2 x dx = \tan x + c$$

$$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$$

$$\int \operatorname{cosec}^2 x dx = -\cot x + c$$

$$\int \sec x \tan x dx = \sec x + c$$

3 Find the following integrals.

- a $\int \sec x (\sec x + \tan x) dx$ b $\int \frac{7x^2 - 5}{x} dx$
 c $\int \operatorname{cosec}^2 x (3e^x \sin^2 x - 5) dx$ d $\int \frac{x^2 - x + 1}{x} dx$

Hint Simplify each expression so it forms one or more standard integral.

4 Evaluate the following. Give your answers as exact values.

- a $\int_{\ln 5}^{\ln 20} e^x dx$ b $\int_2^5 \frac{1 - 3x^3}{x} dx$ c $\int_0^{\frac{\pi}{3}} \left(\frac{\sin x}{\cos^2 x} + 4e^x \right) dx$
 d $\int_0^{\frac{\pi}{4}} \sec^2 x (4 - \cos x \tan x) dx$ e $\int_{\ln 2}^{\ln 6} (e^x + e^{2x}) dx$

(E) 5 Find the exact value of $\int_4^9 \frac{x(1+x)}{x^2} dx$, leaving your answer in the form $a + \ln b$, where a and b are rational numbers. (4 marks)

(E/P) 6 $\int_3^9 \frac{p}{x} dx = \ln 81$, where p is a positive constant. (4 marks)
 Find the value of p .

(E) 7 Find the exact value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x (\cot^2 x - 3e^x \cos^2 x - 1) dx$ (5 marks)

(E/P) 8 Given that a is a real constant, and that

$$\int_{2a}^{5a} \frac{2x+3}{x} dx = \ln 1000$$

show that $a = \ln k$, where k is a constant to be found. (5 marks)

11.2 Integrating $f(ax + b)$

1 a Differentiate:

i $(4x - 5)^4$

ii $(3 - 2x)^6$

b Find:

i $\int 16(4x - 5)^3 dx$

ii $\int (4x - 5)^3 dx$

iii $\int (-12(3 - 2x)^5) dx$

iv $\int 2(3 - 2x)^5 dx$

2 Integrate the following with respect to x .

a $6e^{3x}$

b $\cos(3x + 2)$

c $(4x + 1)^5$

d $\operatorname{cosec} 4x \cot 4x$

e $\sec^2(8x - 1)$

f $\operatorname{cosec}^2(1 - 5x)$

g $\frac{5}{2x - 3}$

h $11e^{4-x}$

3 Find the following integrals.

a $\int \left(\frac{1}{(4x - 3)^2} - (4x - 3)^2 \right) dx$

b $\int \sec^2 3x \left(\frac{1}{2} e^{5x} \cos^2 3x - 2 \right) dx$

c $\int \left(e^{2x} - \frac{1}{e^{2x}} \right)^2 dx$

d $\int \frac{5 - 3 \cos 2x}{\sin^2 2x} dx$

4 Find the exact value of:

a $\int_{-2}^0 \frac{4}{6 - 4x} dx$

b $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \operatorname{cosec}^2 2x (1 - \cos 2x) dx$

c $\int_0^{\ln 4} (e^x - 4)^2 dx$

d $\int_{\frac{5\pi}{8}}^{\frac{3\pi}{4}} \frac{1}{\sin^2(2\pi - x)} dx$

Hint Integration is the reverse process of differentiation, so use your answers to part a to help.

Hint In general,
 $\int f'(ax + b) dx = \frac{1}{a} f(ax + b) + c$

Hint First simplify each expression and then apply the integration rule in question 2 to each term separately.

Hint Integrate each expression and then apply the limits to evaluate the integral.

(E/P) 5 Given that a is a positive constant and $\int_1^a (2x + 1)^2 dx = 117$, find the value of a . **(4 marks)**

(E/P) 6 $\int_1^4 \frac{k}{2x - 1} dx = \ln 25$, where k is a real constant.

Find the value of k .**(4 marks)**

(E/P) 7 Find the exact value of $\int_{\ln 2}^{\ln 8} (e^x + 2)^2 dx$, giving your answer in the form $p + \ln q$, where p and q are integers. **(4 marks)**

(E/P) 8 Given that $\int_{\frac{\pi}{6k}}^{\frac{\pi}{3k}} \frac{\pi}{2} \cos kx dx = \frac{9\pi}{4}(1 - \sqrt{3})$, where k is a real constant, find the value of k .

(5 marks)

11.3 Using trigonometric identities

1 Integrate the following with respect to x .

a $\sin^2 x$ b $3 \cot^2 x$
 c $(1 + \cos x)^2$ d $\frac{\sin^2 x}{\cos^2 x}$

2 Find the exact value of:

a $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos 2x}{\sin^2 x} dx$

b $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin x - \operatorname{cosec} x)^2 dx$

c $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{(1 - \cos x)^2}{\sin^2 x} dx$ d $\int_0^{\frac{\pi}{4}} (\sec x - \tan x)^2 dx$

3 a Using the addition formula, expand:

i $\cos(5x + 2x)$ ii $\cos(5x - 2x)$

b By adding parts a i and a ii together, show that

$$\cos 7x + \cos 3x = 2 \cos 5x \cos 2x$$

c Use your answer to part b to find $\int 4 \cos 5x \cos 2x dx$.

(E/P) 4 Show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 x dx = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$ (4 marks)

(E/P) 5 Show that $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \tan^2 2x dx = \frac{\sqrt{3}}{3} - \frac{\pi}{12}$ (5 marks)

(E/P) 6 a By considering $\sin(6x + x)$ and $\sin(6x - x)$, or otherwise, show that
 $\sin 7x + \sin 5x = 2 \sin 6x \cos x$ (3 marks)

b Hence find $\int 3 \sin 6x \cos x dx$. (3 marks)

Hint Use a trigonometric identity to transform each expression into a sum of standard integrals. ← Sections 6.4, 7.3

Hint In part a, use $\cos 2A \equiv 1 - 2 \sin^2 A$ together with the standard result for integrating $\operatorname{cosec}^2 x$. ← Section 7.3

Hint $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$
 $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$
 ← Section 7.1

11.4 Reverse chain rule

1 a Differentiate:

i $(x^2 - 4)^3$ ii $(x^2 + 4x - 5)^4$

b Find:

i $\int 6x(x^2 - 4)^2 dx$ ii $\int 15x(x^2 - 4)^2 dx$

iii $\int -8(x + 2)(x^2 + 4x - 5)^2 dx$

2 Find the following integrals.

a $\int 6x(x^2 - 7)^{\frac{1}{2}} dx$ b $\int 5 \sin 2x e^{\cos 2x} dx$

c $\int (6x^2 + 8)(4x^3 + 16x - 5)^{\frac{3}{2}} dx$

Hint Integration is the reverse process of differentiation, so use your answers to part a to help.

Hint To integrate an expression in the form $\int k f'(x) (f(x))^n dx$, try differentiating $(f(x))^{n+1}$, then adjust for a constant if necessary.

d $\int \sec^2 x \sqrt{1 + \tan x} dx$ e $\int \operatorname{cosec}^2 3x e^{\cot 3x} dx$ f $\int \cos^4 5x \sin 5x dx$

3 Find the following integrals.

a $\int \frac{4x}{2-x^2} dx$ b $\int \frac{\sin x}{4-3\cos x} dx$

c $\int \frac{6x^2-9}{14+9x-2x^3} dx$ d $\int \frac{\sin 2x}{\cos^2 x} dx$

e $\int \frac{\frac{1}{2} \sin x \cos x}{4 - \cos 2x} dx$ f $\int \frac{e^{-3x}}{1 + e^{-3x}} dx$

4 Find the exact value of:

a $\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \sin^2 4x \cos 4x dx$ b $\int_0^{\frac{\pi}{9}} \frac{\sec^2 3x \tan 3x}{\sec^2 3x} dx$

c $\int_{\ln 2}^{\ln 3} \frac{6e^{3x}}{e^{3x} + 1} dx$

d $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} 3x \cot 3x e^{\operatorname{cosec} 3x} dx$

Hint To integrate an expression in the form $\int k \frac{f'(x)}{f(x)} dx$, try differentiating $\ln(f(x))$, then adjust for a constant if necessary.

Hint Decide carefully whether each expression is in the form $kf'(x)(f(x))^n$ or $k \frac{f'(x)}{f(x)}$. Then integrate and apply the limits to evaluate each integrand.

E/P 5 Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (1 + \cot x)^3 \operatorname{cosec}^2 x dx$, leaving your answer in the form $a + b\sqrt{3}$. (5 marks)

E/P 6 a Show that $\sin^5 x \equiv \sin x - 2 \sin x \cos^2 x + \sin x \cos^4 x$ (3 marks)

b Hence find $\int \sin^5 x dx$. (3 marks)

E 7 Evaluate $\int_0^4 (x+1) \sqrt{2x^2+4x} dx$, leaving your answer in the form $k\sqrt{3}$. (3 marks)

E 8 Find the exact value of $\int_0^{\frac{\pi}{8}} \sin 4x (e^{\cos 4x}) dx$. (4 marks)

E 9 Find the exact value of $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{(3 - \tan x)^4} dx$, leaving your answer as a simplified fraction. (5 marks)

E/P 10 Given that $\int_1^k \frac{x}{x^2+3} dx = \ln \frac{3}{2}$ and $k > 0$, find the exact value of k . (4 marks)

11.5 Integration by substitution

1 Complete the following steps to find $\int 2x(x+2)^4 dx$ using integration by substitution.

a Given $u = x + 2$, find $\frac{du}{dx}$

b Given $u = x + 2$, write $2x$ in terms of u .

c Use your answers to parts a and b to show that $\int 2x(x+2)^4 dx = \int 2(u-2)u^4 du$.

d Show that $\int 2(u-2)u^4 du = \frac{1}{3}u^6 - \frac{4}{5}u^5 + c$

e Use the substitution in part a to write your answer in part d in terms of x .

2 Use the given substitution to find:

a $\int x(x-8)^4 dx$; $u = x - 8$

b $\int \sin x \cos x (1 + \cos x)^5 dx$; $u = 1 + \cos x$

c $\int \operatorname{cosec}^2 x \cot x \sqrt{2 + \cot x} dx$; $u = 2 + \cot x$

d $\int \frac{\sqrt{x^2 + 1}}{x} dx$; $u = \sqrt{x^2 + 1}$

3 Use the given substitution to evaluate:

a $\int_{\frac{1}{2}}^{\frac{5}{4}} x \sqrt{4x-1} dx$; $u = 4x - 1$

b $\int_{16}^{25} \frac{4}{\sqrt{x}(\sqrt{x}-9)} dx$; $u = \sqrt{x}$

c $\int_{\frac{2\pi}{3}}^{\frac{3\pi}{4}} \frac{\sin x}{\cos^3 x} dx$; $u = \cos x$

d $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1 + \sin x} dx$; $u = \sin x$

Hint Use $\frac{du}{dx}$ to replace dx with an expression involving du . Then write the integrand in terms of u and integrate. Remember to use the substitution to write your final answer in terms of x .

E/P 4 Using the substitution $u = 1 + x^2$ or otherwise, find the exact value of $\int_0^2 x^3 \sqrt{1 + x^2} dx$. (6 marks)

E/P 5 Using the substitution $u = e^x$, or otherwise, find the exact value of $\int_{\ln 2}^{\ln 6} \frac{1}{1 + e^x} dx$, leaving your answer in the form $\ln a$, where a is a rational number to be found. (6 marks)

E 6 a Using the substitution $x = \tan \theta$, or otherwise, show that $\int \frac{1}{1 + x^2} dx = \arctan x + c$. (4 marks)

b Hence evaluate $\int_{\frac{\sqrt{3}}{3}}^1 \frac{1}{1 + x^2} dx$. (2 marks)

E/P 7 Using the substitution $u^2 = 2x + 1$ or otherwise, show that the exact value of $\int_0^4 \frac{5x}{\sqrt{2x+1}} dx$ is $\frac{370\sqrt{3} - 20}{21}$. (6 marks)

E/P 8 By using a suitable substitution, or otherwise, find $\int \frac{2 \sin 2x}{1 + \sin x} dx$. (5 marks)

11.6 Integration by parts

1 Find the following integrals.

a $\int x \cos x dx$

b $\int x \sec^2 x dx$

c $\int x e^{2x} dx$

d $\int x \operatorname{cosec}^2 4x dx$

e $\int x e^{-4x} dx$

f $\int x \sin 5x dx$

Hint $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$
In part a, let $u = x$ and $\frac{dv}{dx} = \cos x$.

2 Find the following integrals.

a $\int x \ln x dx$

b $\int x^3 \ln x dx$

c $\int \ln x dx$

Hint In part a, let $u = \ln x$ and $\frac{dv}{dx} = x$.
In part c, write $\int \ln x dx$ as $\int 1 \times \ln x dx$.