

Example 8

Find the coordinates of the points where the line $y = x + 5$ meets the circle $x^2 + (y - 2)^2 = 29$.

$$x^2 + (x + 5)^2 = 29 \Rightarrow x^2 + x^2 + 6x + 9 - 29 = 0$$

$$2x^2 + 6x - 20 \Rightarrow (\div 2) \Rightarrow x^2 + 3x - 10 = 0$$

$$(x + 5)(x - 2) = 0$$

$$x = -5 \xrightarrow{y = x + 5} y = 0 \quad (-5, 0)$$

or

$$x = 2 \quad y = 7 \quad (2, 7)$$

Example 9

Show that the line $y = x - 7$ does not meet the circle $(x + 2)^2 + y^2 = 33$.

$$(x + 2)^2 + (x - 7)^2 = 33 \Rightarrow x^2 + 4x + 4 + x^2 + 4x + 49 - 33 = 0$$

$$2x^2 + 8x + 20 = 0 \quad (\div 2) \quad x^2 + 4x + 10 = 0$$

$$b^2 - 4ac = (4)^2 - 4 \times 1 \times 10$$

$$= 16 - 40$$

$$= -24 < 0 \quad \text{no solutions}$$

7 The line $y = 2x - 2$ meets the circle $(x - 2)^2 + (y - 2)^2 = 20$ at A and B .

a Find the coordinates of A and B .

b Show that AB is a diameter of the circle.

$$c(2, 2)$$

$$r = \sqrt{20}$$

$$r = 2\sqrt{5}$$

(5 marks)

(2 marks)

a) $(x - 2)^2 + (2x - 4)^2 = 20$

$$x^2 - 4x + 4 + 4x^2 - 16x + 16 - 20 = 0$$

$$5x^2 - 20x = 0 \quad (\div 5) \quad x^2 - 4x = 0$$

$$x(x - 4) = 0 \quad x = 0 \quad \text{or} \quad x = 4$$

$$x = 0 \xrightarrow{y = 2x - 2} y = -2 \quad A(0, -2)$$

$$x = 4 \xrightarrow{y = 2x - 2} y = 6 \quad B(4, 6)$$

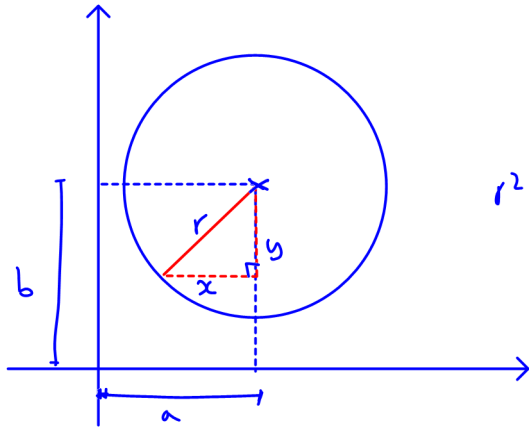
$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(4 - 0)^2 + (6 - (-2))^2}$$

$$= \sqrt{16 + 64}$$

$$= \sqrt{80}$$

$$= 4\sqrt{5} \rightarrow 2r$$



$$r^2 = (x-a)^2 + (y-b)^2$$

$$C(a, b)$$

Example 4

Write down the equation of the circle with centre (5, 7) and radius 4.

$$(x-5)^2 + (y-7)^2 = 4^2$$

$$(x-5)^2 + (y-7)^2 = 16$$

Example 5

A circle has equation $(x-3)^2 + (y+4)^2 = 20$.

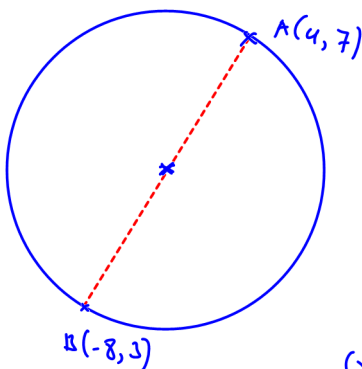
a Write down the centre and radius of the circle.

b Show that the circle passes through (5, -8).

$$\begin{aligned} \text{a) } C(3, -4) \quad r &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned} \quad \begin{aligned} \text{b) } &= (5-3)^2 + (-8+4)^2 \\ &= 2^2 + (-4)^2 \\ &= 4 + 16 \\ &= 20 \text{ on the circle} \end{aligned}$$

Example 6

The line segment AB is a diameter of a circle, where A and B are (4, 7) and (-8, 3) respectively. Find the equation of the circle.



centre is Midpoint

$$C\left(\frac{4-8}{2}, \frac{7+3}{2}\right)$$

$$C(-2, 5)$$

$$(x+2)^2 + (y-5)^2 = r^2 \quad A(4, 7)$$

$$(x+2)^2 + (y-5)^2 = 40$$

$$(4+2)^2 + (7-5)^2 = r^2$$

$$36 + 4 = r^2$$

$$r^2 = 40$$

Find the centre and the radius of the circle with the equation $\underline{x^2 + y^2 - 14x + 16y - 12 = 0}$.

$$(x-7)^2 - 7^2 + (y+8)^2 - 8^2 - 12 = 0$$

$$(x-7)^2 + (y+8)^2 = 125$$

$$C(7, -8) \quad r = \sqrt{125} \\ = 5\sqrt{5}$$

7 The points $P(2, 2)$, $Q(2 + \sqrt{3}, 5)$ and $R(2 - \sqrt{3}, 5)$ lie on the circle $(x - 2)^2 + (y - 4)^2 = r^2$.

a Find the value of r .

(2 marks)

b Show that $\triangle PQR$ is equilateral.

(3 marks)

$$P(2, 2) \rightarrow (2-2)^2 + (2-4)^2 = r^2 \\ 0 + 4 = r^2 \quad (r=2)$$

$$b) |PQ| = \sqrt{(2+\sqrt{3}-2)^2 + (5-2)^2} \quad |PR| = \sqrt{(2-\sqrt{3}-2)^2 + (5-2)^2} \quad |QR| = 2\sqrt{3} \\ = \sqrt{3+9} \quad = \sqrt{3+9} \\ = \sqrt{12} \quad = \sqrt{12} \\ = 2\sqrt{3} \quad = 2\sqrt{3}$$

11 A circle C has equation $x^2 + y^2 + 12x + 2y = k$, where k is a constant.

a Find the coordinates of the centre of C .

(2 marks)

b State the range of possible values of k .

(2 marks)

$$a) (x+6)^2 - 6^2 + (y+1)^2 - 1^2 = k$$

$$(x+6)^2 + (y+1)^2 = k + 37 \quad C(-6, -1)$$

$$b) r = \sqrt{k+37} \quad k+37 \geq 0 \\ k \geq -37$$

13 The circle with equation $(x - k)^2 + y^2 = 41$ passes through the point $(3, 4)$.

Find the two possible values of k .

(5 marks)

$$(3-k)^2 + 4^2 = 41$$

$$(3-k)^2 + 16 = 41 \rightarrow (3-k)^2 = 25 \rightarrow 3-k = \pm 5 \\ \begin{cases} 3-5 = k \Rightarrow k = -2 \\ 3+5 = k \Rightarrow k = 8 \end{cases}$$

- 12 The line with equation $y = 2x + 5$ meets the circle with equation $x^2 + kx + y^2 = 4$ at exactly one point. $\rightarrow b^2 - 4ac = 0$
Find two possible values of k .

$$x^2 + kx + (2x + 5)^2 = 4$$

$$x^2 + \cancel{kx} + 4x^2 + \cancel{20x} + 25 - 4 = 0 \Rightarrow 5x^2 + x(k + 20) + 21 = 0$$

$$(k + 20)^2 - 4 \times 5 \times 21 = 0$$

$$(k + 20)^2 = 420$$

$$k + 20 = \pm 2\sqrt{105} \Rightarrow k = -20 + 2\sqrt{105}$$

or

$$k = -20 - 2\sqrt{105}$$

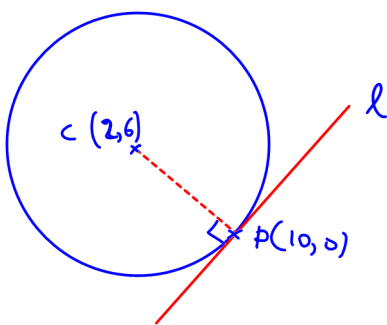
Example 10

The circle C has equation $(x - 2)^2 + (y - 6)^2 = 100$. $C(2, 6)$ $r = 10$

- a Verify that the point $P(10, 0)$ lies on C .
b Find an equation of the tangent to C at the point $(10, 0)$, giving your answer in the form $ax + by + c = 0$.

a) $(10 - 2)^2 + (0 - 6)^2$
 $= 64 + 36$
 $= 100$ on the circle

b)



$$m_{CP} = \frac{0 - 6}{10 - 2} = \frac{-6}{8} = -\frac{3}{4}$$

$$l: m = \frac{4}{3}, P(10, 0)$$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{4}{3}(x - 10)$$

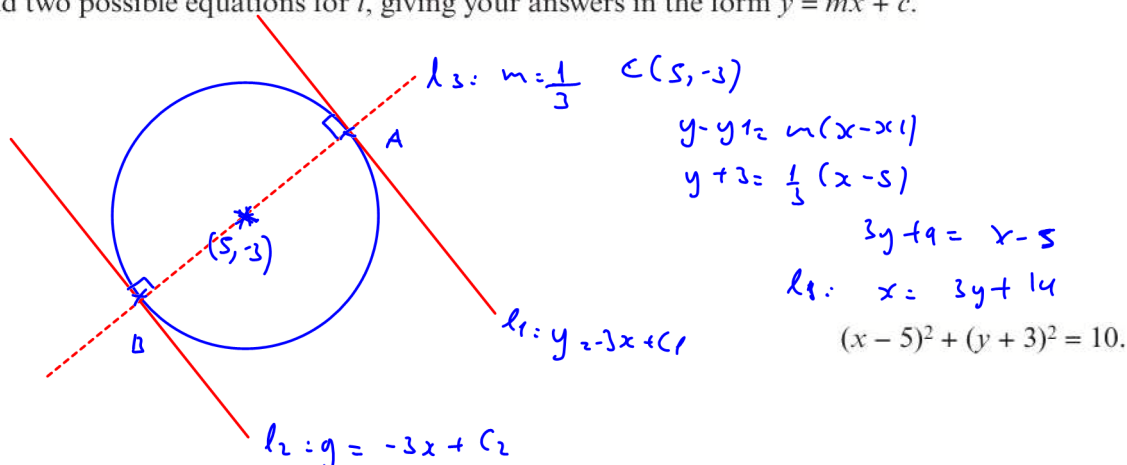
$$3y = 4x - 40 \Rightarrow 4x - 3y - 40 = 0$$

Example 11

A circle C has equation $(x - 5)^2 + (y + 3)^2 = 10$.

The line l is a tangent to the circle and has gradient -3 .

Find two possible equations for l , giving your answers in the form $y = mx + c$.



$$(3y + 14 - 5)^2 + (y + 3)^2 = 10$$

$$(3y + 9)^2 + (y + 3)^2 = 10 \Rightarrow 9y^2 + 54y + 81 + y^2 + 6y + 9 = 10$$

$$10y^2 + 60y + 80 = 0 \div 10$$

$$y^2 + 6y + 8 = 0 \quad (y + 4)(y + 2) = 0$$

$$y = -2 \xrightarrow{x = 3y + 14} x = 8 \quad (8, -2)$$

$$\text{or} \\ y = -4 \quad x = 2 \quad (2, -4)$$

$$l_1: y = -3x + c_1 \Rightarrow -2 = -3 \times 8 + c_1 \Rightarrow c_1 = 22$$

$$l_1: y = -3x + c_2 \Rightarrow -4 = -3 \times 2 + c_2 \Rightarrow c_2 = 2$$

$$l_1: y = -3x + 22$$

$$l_2: y = -3x + 2$$

Example 12

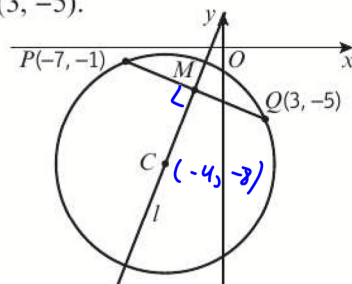
The points P and Q lie on a circle with centre C , as shown in the diagram.

The point P has coordinates $(-7, -1)$ and the point Q has coordinates $(3, -5)$.

M is the midpoint of the line segment PQ .

The line l passes through the points M and C .

a Find an equation for l .



Given that the y -coordinate of C is -8 ,

b show that the x -coordinate of C is -4

c find an equation of the circle.

$$a) M\left(\frac{-7+3}{2}, \frac{-1-5}{2}\right) \Rightarrow M(-2, -3) \quad \text{slope} = \frac{-5+1}{3+7} = \frac{-4}{10} = -\frac{2}{5}$$

$$l: \frac{x}{2} \quad y+3 = -\frac{2}{5}(x+2)$$

$$2y+6 = -2x-4 \Rightarrow 5x-2y+4=0$$

$$b) 5(-2) - 2(-3) + 4 = 0 \Rightarrow 5x+10=0 \Rightarrow x = -\frac{10}{5} \Rightarrow x = -4$$

$$c) (x+4)^2 + (y+8)^2 = r^2 \quad Q(3, -5)$$

$$(3+4)^2 + (-5+8)^2 = r^2$$

$$49+9 = r^2 \Rightarrow r^2 = 58 \quad (x+4)^2 + (y+8)^2 = 58$$

8 The line with equation $2x + y - 5 = 0$ is a

tangent to the circle with equation $(x-3)^2 + (y-p)^2 = 5$ $\rightarrow$ they touch once $\rightarrow$ one solution

a Find the two possible values of p . (8 marks)

$$b^2 - 4ac = 0$$

b Write down the coordinates of the centre of the circle in each case. (2 marks)

$$a) x = \frac{5}{2} - \frac{1}{2}y$$

$$\left(\frac{5}{2} - \frac{1}{2}y - 3\right)^2 + (y-p)^2 = 5$$

$$\left(-\frac{1}{2} - \frac{1}{2}y\right)^2 + (y-p)^2 = 5 \Rightarrow \frac{1}{4} + \frac{1}{2}y + \frac{1}{4}y^2 + y^2 - 2py + p^2 = 5$$

$$1 + 2y + y^2 + 4y^2 - 8py + 4p^2 = 20$$

$$\uparrow \quad \uparrow \quad \uparrow$$

$$5y^2 + (2-8p)y + 4p^2-19 = 0$$

$$b^2 - 4ac = 0 \Rightarrow (2-8p)^2 - 4 \times 5(4p^2-19) = 0$$

$$4 - 32p + 64p^2 - 80p^2 + 380 = 0 \Rightarrow -16p^2 - 32p + 384 = 0 \Rightarrow (-6)$$

$$p^2 + 2p - 24 = 0 \Rightarrow (p+6)(p-4) = 0$$

$$p = -6 \quad \text{or} \quad p = 4$$

$$(x-3)^2 + (y-p)^2 = 5$$

$$(x-3)^2 + (y+6)^2 = 5 \quad C(-3, -6)$$

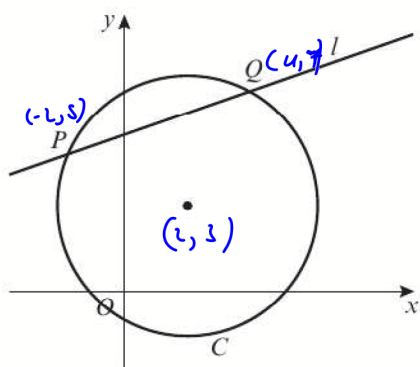
$$(x-3)^2 + (y-4)^2 = 5 \quad C(3, 4)$$

11 The circle C has equation $x^2 - 4x + y^2 - 6y = 7$. $\rightarrow (x-2)^2 + (y-3)^2 = 20$

The line l with equation $x - 3y + 17 = 0$ intersects the circle at the points P and Q .

a Find the coordinates of the point P and the point Q .

(4 marks)



$$x = 3y - 17$$

$$(3y-17)^2 - 4(3y-17) + y^2 - 6y = 7$$

$$9y^2 - 102y + 289 - 12y + 68 + y^2 - 6y - 7 = 0$$

$$10y^2 - 120y + 350 = 0 \quad (\div 10)$$

$$y^2 - 12y + 35 = 0 \quad (y-5)(y-7) = 0$$

$$y = 5, y = 7 \quad x = 3y - 17$$

$$y = 5 \quad x = -2 \quad (-2, 5)$$

$$y = 7 \quad x = 4 \quad (4, 7)$$