

06.10

- 3 Prove that, when n is an odd integer, $3n^2 - 12$ is always odd, and when n is an even integer, $3n^2 - 12$ is always even.

$$\begin{aligned}n = 2h+1 \quad 3(2h+1)^2 - 12 &= 3(4h^2 + 4h + 1) - 12 \\&= 12h^2 + 12h + 3 - 12 \\&= 12h^2 + 12h - 9 \\&= 12h(h+1) - 9 \\&\quad \underbrace{\hspace{1cm}}_{\text{even} \times \text{odd} \Rightarrow \text{odd}} - 9 \Rightarrow \text{odd}\end{aligned}$$

$$\begin{aligned}n = 2h \quad 3(2h)^2 - 12 &= 12h^2 - 12 \\&= 12(h^2 - 1) \\&\quad \text{even}\end{aligned}$$

- 4 Find a counter example to disprove the following statement:

If $x > y$, then $\frac{x}{y} > \frac{y}{x}$.

$$x=1 \quad \text{and} \quad y=-1 \quad \frac{1}{-1} = -1 < 1$$

- 5 Prove that $n^3 + 2n^2 + 12n$ always has a factor of 8 when n is even.

$$\begin{aligned}n = 2h \quad 8h^3 + 8h^2 + 24h &= 8h(h^2 + h + 3) \\&\quad \downarrow \\&\quad \text{factor of 8}\end{aligned}$$

- 6 Prove that $x(x + 2y + 4) \geq 4x - y^2$ for any integers x and y .

$$x^2 + 2xy + \cancel{4x} - \cancel{4x} + y^2$$

$$x^2 + 2xy + y^2 = (x+y)^2 \geq 0 \quad \text{always}$$

- 7 Prove that the difference between the cube and square of any integer is always even.

$$n^3 - n^2 = n^2(n-1)$$

• $n \rightarrow \text{even}$

$$\begin{aligned}n = 2h \quad (2h)^2(2h-1) &\Rightarrow 4h^2(2h-1) \\&\quad \text{even} \times \text{odd} \rightarrow \text{even}\end{aligned}$$

• $n \rightarrow \text{odd}$

$$n = 2h+1 \quad (2h+1)^2(2h+1-1) \Rightarrow \text{odd} \times \text{even} \rightarrow \text{even}$$

8 Prove that the product of any two distinct prime numbers has exactly four factors.

Assume a & b are prime

$a \rightarrow$ factors 1 & a

$b \rightarrow$ factor 1 & b

$ab \rightarrow$ factors $1, a, b, ab$ 4 factors

4 Prove that all cube numbers are either a multiple of 9 or 1 more or 1 less than a multiple of 9. (4 marks)

Assume n is even

$$n = 2h \quad n^3 = (2h)^3 = 8h^3 \quad \text{one less than multiple of 9}$$

$$\begin{aligned} n = 2h+1 \quad n^3 &= (2h+1)^3 \\ &= 8h^3 + 3(2h)^2 + 3(2h) + 1 \\ &= 8h^3 + 12h^2 + 6h + 1 \end{aligned}$$

$$(a+b)^0 = 1$$

$$(a+b)^1 = 1a + 1b$$

$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & 1 & & 1 \\ & & & 1 & & 2 & & 1 \\ & 1 & & 3 & & 3 & & 1 \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$\begin{aligned} n = 2h+1 \quad n^3 &= (2h+1)^3 \\ &= 8h^3 + 3(2h)^2 + 3(2h) + 1 \\ &= 8h^3 + 12h^2 + 6h + 1 \\ &= 2h(4h^2 + 6h + 3) + 1 \end{aligned}$$

X

Assume consecutive numbers

$$3n, 3n+1, 3n+2$$

$$(3n)^3 = 27n^3 \quad \text{multiple of 9}$$

$$(3n+1)^3 = 27n^3 + 27n^2 + 9n + 1$$

multiple of 9 + 1

$$(3n+2)^3 = 27n^3 + 54n^2 + 36n + 8 \rightarrow \text{multiple of 9} - 1$$

10 $g(x) = 2x^3 + 9x^2 - 6x - 5$.

a Factorise $g(x)$.

(6 marks)

b Solve $g(x) = 0$.

(2 marks)

$$g(1) = 2 \times 1^3 + 9 \times 1^2 - 6 \times 1 - 5$$

$$= 0$$

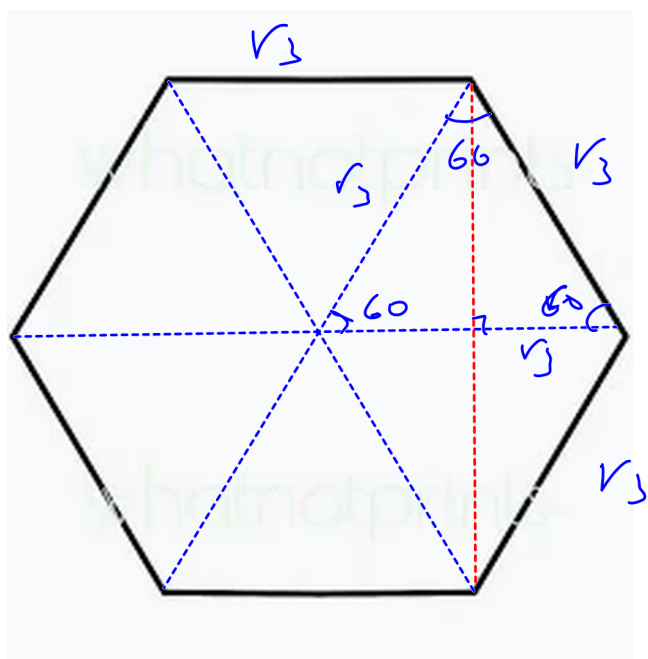
$\therefore x-1$ is a solution $x-1 \rightarrow$ factor

$$\begin{array}{r} x-1 \overline{) 2x^3 + 9x^2 - 6x - 5} \\ \underline{2x^3 - 2x^2} \\ 11x^2 - 6x - 5 \\ \underline{11x^2 - 11x} \\ 5x - 5 \\ \underline{5x - 5} \\ 0 \end{array}$$

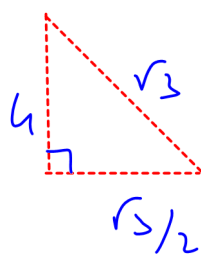
$$(x-1)(2x^2 + 11x + 5)$$

$$(x-1)(2x-1)(x+5)$$

23 Prove that the distance between opposite edges of a regular hexagon of side length $\sqrt{3}$ is a rational value.



$$d = 2h$$



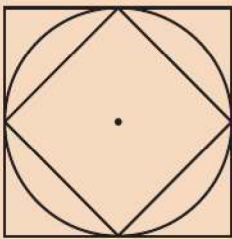
$$h^2 = (\sqrt{3})^2 - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$h^2 = 3 - \frac{3}{4}$$

$$h^2 = \frac{9}{4} \Rightarrow h = \frac{3}{2}$$

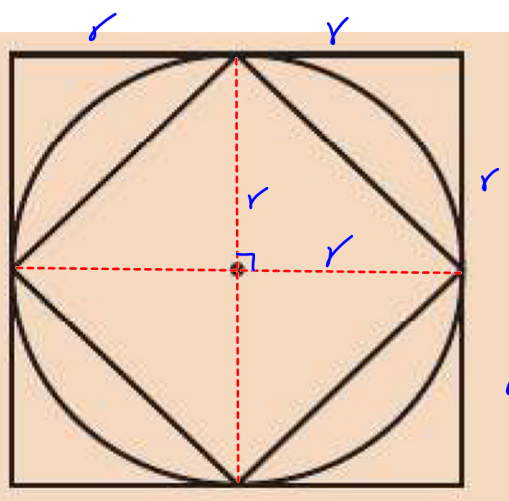
$$2h = 3$$

1 The diagram shows two squares and a circle.



a Given that π is defined as the circumference of a circle of diameter 1 unit, prove that $2\sqrt{2} < \pi < 4$.

we need to use the fact
that the perimeter of the circle
is somewhere between the perimeter
of large square and the small one
circle = $2\pi r$



$$P_{\text{large}} = 4(2r) = 8r$$

$$\begin{aligned} l^2 &= r^2 + r^2 \\ l &= r\sqrt{2} \end{aligned}$$

$$\begin{aligned} P_{\text{small}} &= 4l \\ &= 4r\sqrt{2} \end{aligned}$$

$$P_{\text{small}} < P_{\text{circle}} < P_{\text{large}} \\ 4r\sqrt{2} < 2\pi r < 8r \quad (\div 2r)$$

$$2\sqrt{2} < \pi < 4$$

2 Prove that if $f(x) = ax^3 + bx^2 + cx + d$ and $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$.

$$\frac{f(x)}{x-p} = g(x)$$

$$f(x) = (x-p) g(x)$$

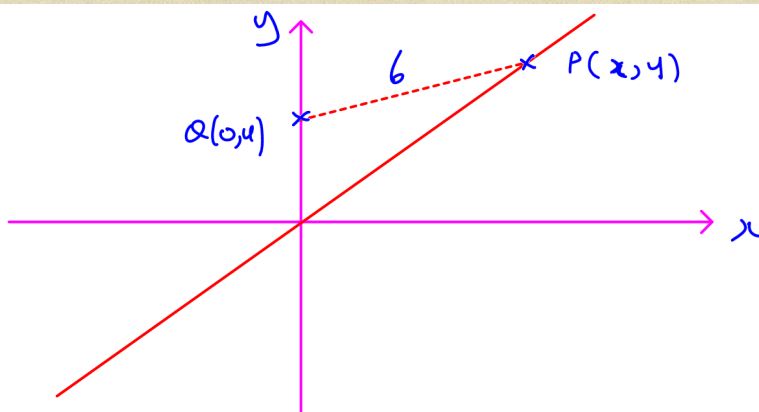
$$f(p) = \frac{0}{\cancel{p-p}} g(x)$$

$$= 0$$

The point P has a positive x -coordinate and lies on the line with equation $y = x$.

The point Q has coordinates $(0, 4)$. Given that the distance PQ is 6, find the exact coordinates of P .

(8 marks)



$$\sqrt{(x-0)^2 + (y-4)^2} = 6 \Rightarrow (x-0)^2 + (y-4)^2 = 36$$

since P is on $y=x$ let $x=y$

$$(y-0)^2 + (y-4)^2 = 36$$

$$y^2 + y^2 - 8y + 16 - 36 = 0$$

$$2y^2 - 8y - 20 = 0 \Rightarrow (\div 2) \quad y^2 - 4y - 10 = 0$$

$$y = 2 + \sqrt{14} \quad y = 2 - \sqrt{14}$$

$$(2 + \sqrt{14}, 2 + \sqrt{14})$$

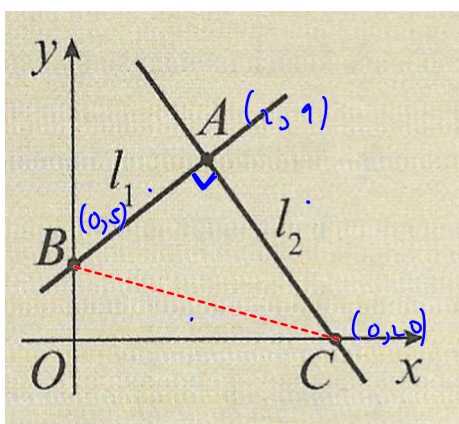
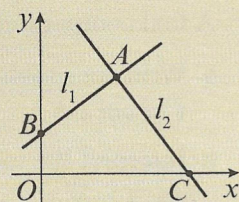
Gold

The points $A(2, 9)$ and $B(0, 5)$ lie on the line l_1 , as shown in the diagram.

The line l_2 is perpendicular to l_1 , passes through A and crosses the x -axis at the point C .

Find the area of $\triangle ABC$.

(8 marks)



$$\text{Area} = \frac{1}{2} |AB| \times |AC|$$

$$\frac{l_2}{m_{AB}} = \frac{9-5}{2-0} = 2$$

$$l_2: m = -\frac{1}{2} \quad (1, 1)$$

$$y - 9 = -\frac{1}{2}(x - 2) \quad y = 0$$

$$-9 = -\frac{1}{2}(x - 2) = -\frac{1}{2}x + 1 \quad x = 10$$

$$\frac{1}{2} |AB| |AC| \sin 90^\circ$$