

9 Differentiation

9.1 Differentiating $\sin x$ and $\cos x$

1 Find $\frac{dy}{dx}$ given that:

a $y = 6 \sin x$ b $y = -3 \sin \frac{1}{2}x$ c $y = \frac{1}{3} \sin 5x$
 d $y = \cos 4x$ e $y = 5 \cos 6x$ f $y = -\frac{1}{2} \cos 7x$

Hint

If $y = \sin kx$, then $\frac{dy}{dx} = k \cos kx$

If $y = \cos kx$, then $\frac{dy}{dx} = -k \sin kx$

2 Find $f'(x)$ given that:

a $f(x) = 4 \sin x - 3 \cos x$
 b $f(x) = \frac{1}{2} \sin 2x + 4 \cos x - 6 \sin \frac{1}{4}x$
 c $f(x) = 5x^{\frac{3}{2}} - \frac{4}{x^2} - 6 \sin \frac{1}{3}x$ d $f(x) = \frac{8}{\sqrt{x}} + 3 \cos \frac{1}{6}x$

Hint

Differentiate each term separately.

3 $f(x) = 5 \sin x - 3 \cos x$. Find the x -coordinates of the stationary points on the curve with equation $y = f(x)$ in the interval $-180^\circ \leq x \leq 180^\circ$.

Hint

Stationary points occur when the gradient is equal to zero.

4 $f(x) = x - \cos 2x$. Find the equation of the tangent to the curve at the point $(\frac{\pi}{4}, \frac{\pi}{4})$.

Hint

First differentiate to find the gradient, m . Then find the equation of the tangent using the equation $y - y_1 = m(x - x_1)$

← Year 1, Section 12.6

E 5 $f(x) = 6 \sin x - 3 \cos 2x$
 Find the stationary points of the curve $y = f(x)$ in the interval $0 \leq x \leq 2\pi$. (6 marks)

E/P 6 $g(x) = 6 \sin 2x - 4 \cos 2x$. Find the equation of the normal to the curve with equation $y = g(x)$ at the point $(\pi, -4)$. (5 marks)

E/P 7 $h(x) = 3 \sin x - 4 \cos x + \frac{1}{2}x^2$. Show that the equation of the tangent to the curve with equation $y = h(x)$ at the point with x -coordinate $\frac{\pi}{2}$ is $8y - (32 + 4\pi)x = 24 - 16\pi - \pi^2$. (6 marks)

E/P 8 Prove from first principles that the derivative of $\cos x$ is $-\sin x$. (5 marks)

9.2 Differentiating exponentials and logarithms

1 Differentiate:

a $6e^{-3x}$ b $5e^{2x} - 7e^{-x}$
 c $\frac{7}{e^{5x}}$ d $\frac{(1 - e^{2x})^2}{e^{2x}}$

Hint

If $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx}$

2 Find $f'(x)$ given that:

a $f(x) = \ln 2x$ b $f(x) = \ln x^2$
 c $f(x) = \ln \left(\frac{7}{x}\right)$ d $f(x) = \ln(4x^3)$

Hint

If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

First expand the expression using laws of logarithms. ← Year 1, Section 14.5

3 Find $\frac{dy}{dx}$ given that:

a $y = 3^x$

b $y = 5^{2x}$

c $y = \left(\frac{2}{5}\right)^{4x}$

d 8^{-x}

Hint

If $y = a^{kx}$, where k is a real constant and $a > 0$, then $\frac{dy}{dx} = a^{kx} k \ln a$.

4 Find the equation of the tangent to the curve $y = 4^x$ at the point with x -coordinate $-\frac{1}{2}$

(E) 5 Given $f(x) = e^{-x} + 2 \ln x^3$, find $f'(x)$. (3 marks)

(E/P) 6 A curve has equation $y = 3e^x - \ln x^2$. Show that the equation of the normal at the point with x -coordinate 1 is $x + (3e - 2)y - 9e^2 + 6e - 1 = 0$. (6 marks)

(E/P) 7 $f(x) = 4^x + 4^{-x}$. Show that the equation of the tangent to the curve with equation $y = f(x)$ at the point with x -coordinate $\frac{1}{2}$ is $y = (\ln 8)x + \frac{5}{2} - \frac{1}{2} \ln 8$. (6 marks)

(E/P) 8 $f(x) = 9^{3x} + 3^{6x}$

a Demonstrate, clearly showing all your steps, that $f(x)$ can be written in the form $a(3^{kx})$ where a and k are constants to be found. (2 marks)

b Hence, or otherwise, show that $f'(x) = (4 \ln 3)(3^{6x+1})$ (3 marks)

9.3 The chain rule

1 Find $\frac{dy}{dx}$ for each of the following:

a $y = (4 - 3x)^3$

b $y = (6 + 5x)^{\frac{1}{2}}$

c $y = (5 + 2x^3)^4$

d $y = (7 + 3x^2)^{-5}$

e $y = \frac{6}{\sqrt{4 - 5x}}$

f $y = \frac{1}{6 - x}$

g $y = \sqrt[3]{3 + 2x^4}$

h $y = \sqrt{7x^3 - 4}$

2 Differentiate:

a $y = \sin^3 x$

b $y = \ln x^4$

c $y = e^{\cos 4x}$

d $y = (\ln x)^2$

e $y = \ln(\cos x)$

f $y = \sin(e^{-3x})$

3 $f(x) = (3 - \ln 6x)^2$

a Find $f'(x)$.

b Evaluate $f'\left(\frac{e^2}{6}\right)$.

Hint

If $y = (f(x))^n$, then $\frac{dy}{dx} = n(f(x))^{n-1}f'(x)$.

Hint

You can use the chain rule to differentiate a function of a function.

If $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x))g'(x)$.

- 4 Find the value of $\frac{dy}{dx}$ at the point (3, 6) on the curve with equation $2y^2 - 4y = x$.

Hint

First find $\frac{dx}{dy}$
Then use the fact that $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

- E/P** 5 A curve has equation $y = (2x - 3)^5$. Find the equation of the normal to the curve at the point P with x -coordinate 2. (5 marks)

- E/P** 6 $f(x) = \frac{3}{(6-x)^3}$, $x \neq 6$. Find the tangent to the curve with equation $y = f(x)$ at the point P with x -coordinate 4. Give your answer in the form $ax + by + c = 0$. (7 marks)

- E/P** 7 $f(x) = 3\sin^2 x - 2\cos^2 x$. Show that the equation of the tangent to the curve with equation $y = f(x)$ at the point Q with x -coordinate $-\frac{5\pi}{6}$ is $12y - 30\sqrt{3}x - 25\pi\sqrt{3} + 9 = 0$. (6 marks)

- E/P** 8 The curve C has equation $x = -2\sin 4y$. Show that the equation of the normal to C at the point A with y -coordinate $\frac{2\pi}{3}$ is $4x + y + 4\sqrt{3} - \frac{2\pi}{3} = 0$. (6 marks)

- E/P** 9 Find the equation of the tangent to the curve with equation $y = 2^{(x^2)}$ at the point with coordinates $(-1, 2)$. Give your answer in the form $ax + by + c = 0$, where a , b and c are exact values to be found. (6 marks)

9.4 The product rule

- 1 Find $\frac{dy}{dx}$ for each of the following:

a $y = x(4 - 5x)^3$

b $y = 3x^3(2x - 1)^6$

c $y = \frac{4}{x^2}(6x - 7)^4$

d $y = 8x^2(2x^3 - 4)^{-3}$

Hint

If $y = uv$, then $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$,
where u and v are functions of x .

- 2 Find $f'(x)$ given that:

a $f(x) = x^2 \cos 2x$

b $f(x) = e^{3x} \sin 5x$

c $f(x) = e^{-x}(2x^2 + 3)^3$

d $f(x) = -3 \cos 4x \sin 2x$

e $f(x) = x^2 \ln x$

f $f(x) = e^{4x} \ln(\cos 3x)$

Hint

If $f(x) = g(x)h(x)$ then
 $f'(x) = g(x)h'(x) + h(x)g'(x)$

- 3 Find the value of $\frac{dy}{dx}$ at the point (1, 4) on the curve with equation $y = 4x^3(1 - 2x^2)^4$.

Hint

Evaluate $\frac{dy}{dx}$ when $x = 1$.

- 4 Differentiate with respect to x :

a $(x^4 - 3x^2)e^{-2x}$

b $(4 \cos 2x - \sin x)e^{3x}$

c $(x^2 - 5 \cos \frac{1}{2}x) \ln 4x$

- E** 5 Differentiate with respect to x :

a $e^{-x} \cos 3x$ (2 marks)

b $x^4 \ln(\sin x)$ (2 marks)

- E/P** 6 a A curve C has equation $y = 2x^4(2 - 5x^2)^3$. Show that $\frac{dy}{dx} = x^3(2 - 5x)^n(ax^2 + bx + c)$, where n, a, b and c are constants to be found. (4 marks)
- b Hence find the exact x -coordinates of the stationary points of C . (4 marks)
- E/P** 7 The curve C has equation $f(x) = e^{4x} \sin x$.
- a Show that the turning points of C occur when $\tan x = -\frac{1}{4}$. (3 marks)
- b Show that the equation of the normal to C at the point where $x = \frac{\pi}{2}$ is $x + 4e^{2\pi}y - 4e^{4\pi} - \frac{\pi}{2} = 0$. (3 marks)
- E/P** 8 A curve has equation $y = (x - \pi) \sin 2x$. Show that the equation of the tangent to the curve at the point with x -coordinate $\frac{3\pi}{4}$ is $x + y = \pi$. (6 marks)

9.5 The quotient rule

- 1 Find $f'(x)$ given that:

a $f(x) = \frac{4x+3}{2x-9}$

b $f(x) = \frac{6x^2}{3-5x}$

c $f(x) = \frac{1-8x}{(3x^3+2)^2}$

d $f(x) = \frac{7x^2-9x}{\sqrt{x^2+6}}$

Hint

If $f(x) = \frac{g(x)}{h(x)}$ then

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$$

- 2 Find $\frac{dy}{dx}$ for each of the following:

a $y = \frac{4x^2}{\cos 2x}$

b $y = \frac{x - e^{-x}}{\ln x}$

c $y = \frac{e^x}{\cos^2 x}$

d $y = \frac{5 - e^{-x}}{\sin 4x}$

Hint

If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$ where u and v are functions of x .

- 3 Find the gradient of the curve with equation $y = \frac{8-2x}{4x+1}$ at the point $(\frac{1}{2}, \frac{7}{3})$
- 4 The curve C has equation $y = \frac{x^2-6x}{e^{2x}}$
- a Find $\frac{dy}{dx}$. b Find the equation of the normal to C at $(1, -\frac{5}{e^2})$.

- E** 5 Given $y = \frac{\sin^2 x}{e^{2x}}$, find $\frac{dy}{dx}$. (4 marks)

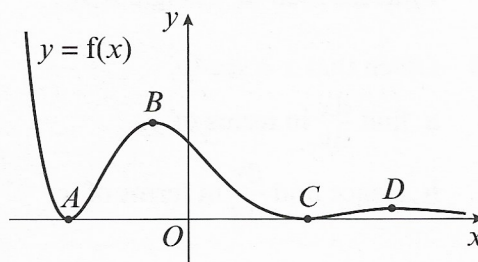
- E** 6 Differentiate $\frac{e^{4x}}{x(x-5)}$ with respect to x . (4 marks)

- E/P** 7 The curve C has equation $x = \frac{\cos 2y}{e^y}$. Find the equation of the tangent to the curve at the point where $y = \frac{\pi}{8}$ in the form $ax + by + c = 0$. (6 marks)

- E/P** 8 The diagram shows part of the graph with equation $y = f(x)$, where $f(x) = \frac{\cos^2 x}{e^x}$

- a Show that at the stationary points A, B, C and D , either $\cos x = 0$ or $\tan x = -\frac{1}{2}$. (4 marks)

- b Hence, or otherwise, find the x -coordinates of the points A, B, C and D , giving your answers in radians to 2 decimal places. (2 marks)



9.6 Differentiating trigonometric functions

1 Find $\frac{dy}{dx}$ given:

a $y = \tan 4x$

b $y = 7 \cot 2x$

c $y = \cot^5 x$

d $y = -3 \tan^3 x$

2 Find $\frac{dy}{dx}$ given:

a $y = \operatorname{cosec} 6x$

b $y = -4 \sec 5x$

c $y = (\operatorname{cosec} x)^{\frac{4}{3}}$

d $y = -2 \sec^3 4x$

3 Find $f'(x)$ when $f(x)$ is:

a $x^2 \tan 3x$

b $\frac{e^{\sec x}}{x^2 + 1}$

c $\ln(\operatorname{cosec}^2 x)$

d $e^{-x} \cot 4x$

e $\frac{\sec^2 x}{\ln x}$

f $\frac{(4x-1)^3}{\cot^2 x}$

4 Given that $x = \sin 2y$,

a find $\frac{dx}{dy}$ in terms of y .

b Hence find $\frac{dy}{dx}$ in terms of x .

(E) 5 Show that if $y = \operatorname{cosec} x$, then $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$.

(5 marks)

(E/P) 6 $f(x) = \ln(\cot 2x)$

a Show that $f'(x) = -4 \operatorname{cosec} 4x$.

(5 marks)

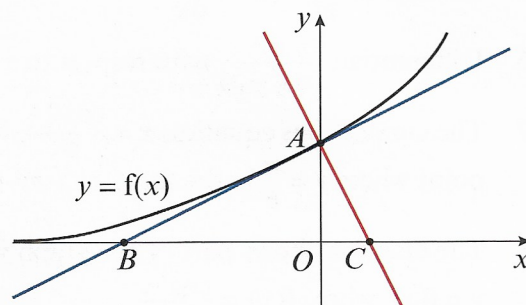
b Find the equation of the normal to the curve $y = f(x)$ at the point with x -coordinate $\frac{\pi}{8}$

(3 marks)

(E/P) 7 The diagram shows part of the curve $y = f(x)$, where $f(x) = e^{\tan^{\frac{1}{2}} x}$

Point A is the y -intercept of $y = f(x)$. The tangent and normal to the curve $y = f(x)$ at A intersect the x -axis at the points B and C respectively. Find the area of triangle ABC .

(8 marks)



(E/P) 8 Given that $x = \sec 4y$,

a find $\frac{dx}{dy}$ in terms of y .

(2 marks)

b Hence find $\frac{dy}{dx}$ in terms of x .

(4 marks)

Hint

If $y = \tan kx$, then $\frac{dy}{dx} = k \sec^2 kx$

If $y = \cot kx$, then $\frac{dy}{dx} = -k \operatorname{cosec}^2 kx$

Hint

If $y = \operatorname{cosec} kx$, then

$\frac{dy}{dx} = -k \operatorname{cosec} kx \cot kx$

If $y = \sec kx$, then $\frac{dy}{dx} = k \sec kx \tan kx$

Hint

Use the product rule or quotient rule in addition to the chain rule and standard rules for differentiating trigonometric functions.

Hint

Use the identity $\sin^2 A + \cos^2 A = 1$ to write $\cos 2y$ in terms of x .

← Year 1, Section 10.3

9.7 Parametric differentiation

- 1 Find $\frac{dy}{dx}$ for each of the following, leaving your answers in terms of the parameter t .

a $x = 4t - 1, y = 2t^3$

b $x = \frac{2}{t^2}, y = 4 - t^2$

c $x = 3t, y = te^{4t}$

d $x = 4 \sin t, y = 5 \tan t$

e $x = 4e^{3t} - 1, y = 2e^t + 4$

f $x = 2t^3 - 1, y = \ln t$

Hint

If x and y are given as functions of a

parameter, t , then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

- 2 A curve C has parametric equations $x = \tan t, y = \sec t$. Find:

a $\frac{dy}{dx}$ in terms of the parameter t

- b the equation of the tangent to the curve at the point P , where $t = \frac{\pi}{4}$

Hint

Substitute $t = \frac{\pi}{4}$ into the original equations to find the coordinates of P .

- 3 A curve C has parametric equations $x = 1 - t^2, y = t^3 + 2t^2$. Find:

a $\frac{dy}{dx}$ in terms of the parameter t

- b the equation of the normal to the curve at the point P , where $t = -3$.

Give your answer in the form $ax + by + c = 0$, where a, b and c are constants to be found.

- 4 A curve C has parametric equations $x = 1 - \cos 2\theta, y = 4 \sin 2\theta, -\pi \leq \theta \leq \pi$. Find:

a $\frac{dy}{dx}$ in terms of the parameter θ

- b the coordinates of the points with zero gradient on the curve C .

Hint

Solve $\frac{dy}{dx} = 0$ to find the values of θ at the points with zero gradient, then substitute into the original parametric equations to find the coordinates.

- E/P** 5 The curve C has parametric equations $x = 2 \sec \theta, y = 4 \cos^2 \theta, -\pi \leq \theta \leq \pi$.

a Find an expression for $\frac{dy}{dx}$ in terms of the parameter θ . (4 marks)

b Find an equation of the normal to the curve at the point where $\theta = \frac{\pi}{3}$ (4 marks)

- E/P** 6 The curve C has parametric equations $x = (t - 2)^2, y = t^2 + 4t, t \in \mathbb{R}$.

The line l is a tangent to C and is parallel to the line with equation $y = 2x - 9$.

Find the equation of l . (6 marks)

- E/P** 7 The curve C has parametric equations $x = -2 \sec 2t, y = 2 \cot t, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

a Find an expression for $\frac{dy}{dx}$ in terms of the parameter t . (4 marks)

The point A with coordinates $(-4, -2\sqrt{3})$ lies on the curve C . The line l is tangent to C at A .

b Show that an equation for l is $x\sqrt{3} + 3y + 10\sqrt{3} = 0$. (4 marks)

- E/P** 8 The curve C has parametric equations $x = \frac{t}{1-t^2}$, $y = \frac{1}{1+t^2}$, $-1 < t < 1$.
The line l is normal to C at the point where $t = -\frac{1}{2}$

a Find an equation for the line l . (6 marks)

b Show that a Cartesian equation for the curve C is $y = \frac{x^2 + 2x + 1}{2x^2 + 2x + 1}$. (4 marks)

9.8 Implicit differentiation

- 1 By writing $u = y^5$ and using the chain rule, show that $\frac{d}{dx}(y^5) = 5y^4 \frac{dy}{dx}$.

Hint Use $\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$

- 2 Show that $\frac{d}{dx}(x^3y^2) = 2x^3y \frac{dy}{dx} + 3x^2y^2$

Hint Use the product rule and the fact that $\frac{d}{dx}(y^n) = ny^{n-1} \frac{dy}{dx}$

- 3 Find $\frac{dy}{dx}$ in terms of x and y where:

a $x^4 - y^3 = 3$

b $3x^2 + 4y^2 - 5x^2y = 2$

c $y = \frac{5y^2}{x^2 - y^2}$

d $4 \ln(x-5) - \ln(y^3) = 0$

e $\cos x + \sin^2 y = 1$

f $xe^{-y} + ye^{2x} - 7 = 0$

Hint When you differentiate implicitly, your expression for $\frac{dy}{dx}$ will usually involve both x and y .

- 4 The curve C is described by the equation $x^2 - 4x^2y + y^2 = 21$

a Find $\frac{dy}{dx}$

b Evaluate $\frac{dy}{dx}$ at the point $(2, -1)$.

c Find an equation of the tangent to C at the point $(2, -1)$.

Hint For part b, substitute $x = 2$ and $y = -1$ into your expression for $\frac{dy}{dx}$

- E** 5 Find the gradient of the curve with equation $3x^2 - y^3 - 5xy = 1$ at the point $(2, 1)$. (5 marks)

- E** 6 Given that $e^{-3y} - e^{-2x} = 4xy$, find $\frac{dy}{dx}$ in terms of x and y . (5 marks)

- E/P** 7 The curve C satisfies $\cos 2x + \sin 2y = 1$, where $-\frac{\pi}{4} < x < \frac{\pi}{4}$ and $0 < y < \frac{\pi}{6}$

a Find an expression for $\frac{dy}{dx}$. (3 marks)

b Find an equation of the tangent to C at the point P with x -coordinate $-\frac{\pi}{6}$, giving your answer in the form $ax + by + c = 0$, where a , b and c are constants to be found. (5 marks)

- E/P** 8 A curve C is described by the equation $4x^2 + y^2 + 6x - 8y - 10 = 0$.
Find an equation of the normal to C at the point $(1, 8)$, giving your answer in the form $ax + by + c = 0$, where a , b and c are constants to be found. (7 marks)

9.9 Using second derivatives

1 $f(x) = e^{-x} \cos 2x$

- a Find $f'(x)$. b Find $f''(x)$.

Hint

To find $f''(x)$, differentiate $f'(x)$.

2 $f(x) = 8x^2 + 3e^{4x}$

- a Find $f''(x)$.
b Hence show that $f(x)$ is a convex function for all real values of x .

Hint

The function $f(x)$ is convex on the interval $[a, b]$ if and only if $f''(x) \geq 0$ for every value of x in that interval.

3 $f(x) = \cos 2x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

- a Find $f''(x)$.
b Find the range of values for which $f(x)$ is concave.

Hint

The function $f(x)$ is concave on the interval $[a, b]$ if and only if $f''(x) \leq 0$ for every value of x in that interval.

4 $f(x) = x(2x - 1)^5$

- a Find $f'(x)$ and $f''(x)$.
b Show that $f''(x) = 0$ at $x = \frac{1}{6}$ and $x = \frac{1}{2}$.
c Show that $x = \frac{1}{6}$ is a point of inflection by demonstrating that $f''(0.1)$ and $f''(0.2)$ have different signs.

Hint

A point of inflection is a point at which $f''(x)$ changes sign. To find the point of inflection, solve $f''(x) = 0$, then check that $f''(x)$ has different signs on either side of that point.

- (E)** 5 Find the interval on which the function $f(x) = x^3 - 8x^2 + 6x - 2$ is convex. **(4 marks)**

- (E)** 6 $f(x) = x(2x - 5)^4$
a Find $f'(x)$ and $f''(x)$. **(4 marks)**

- b Show that $f(x)$ has a point of inflection at $x = 1$. **(3 marks)**

- (E/P)** 7 The curve C has equation $y = f(x)$ where $f(x) = \frac{\sin 4x}{e^{2x}}, 0 \leq x \leq \frac{\pi}{4}$.
Determine the range of values for which $f(x)$ is convex. Give your answer to 2 decimal places where necessary. **(6 marks)**

- (E/P)** 8 $f(x) = x^3 e^x$. Find the exact range of values for which the curve C with equation $y = f(x)$ is concave. **(6 marks)**

9.10 Rates of change

1 Given that $S = 6x^2$ and that $\frac{dx}{dt} = \frac{2}{3}$,

- a find $\frac{dS}{dt}$
b evaluate $\frac{dS}{dt}$ when $x = 3$.

Hint

You can use the chain rule to connect rates of change in situations involving two or more variables. Here $\frac{dS}{dt} = \frac{dS}{dx} \times \frac{dx}{dt}$

2 Given that $y = x \cos 4x$ and that $\frac{dx}{dt} = 3$, find $\frac{dy}{dt}$ when $x = \frac{5\pi}{16}$

3 Given that $y = \frac{e^{2x}}{x^2}$ and that $\frac{dx}{dt} = -\frac{1}{2}$, find $\frac{dy}{dt}$ when $x = 4$.

4 The volume of a sphere $V \text{ cm}^3$ is related to its radius by the formula $V = \frac{4}{3}\pi r^3$.

a Find $\frac{dV}{dr}$

b Given that the rate of change of the radius in cm s^{-1} is given by $\frac{dr}{dt} = 3$, find $\frac{dV}{dt}$

Hint

For part **b**, r varies with t , so you cannot treat it as a constant when you differentiate V with respect to t . You need to use the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

E 5 At any given time, the rate of population growth in a small town is modelled as being proportional to the current population, P .

a Write a differential equation to describe this model. (1 mark)

b Explain how this model could be adapted if the population is declining at a rate proportional to its size. (1 mark)

E 6 At time t years after monitoring began, the area covered by Arctic ice is $I \text{ km}^2$.

The rate of decay of ice coverage is directly proportional to the square root of the existing area covered. Write down a differential equation in terms of I and t . (2 marks)

E 7 The value, V in £s, of a car t years after it is purchased is modelled by the equation

$$V = 23500e^{-0.25t}, t \geq 0$$

a Find the value of the car 7 years after it was purchased. (1 mark)

b Find the rate of change of value of the car after 10 years. (3 marks)

E 8 The volume, $V \text{ cm}^3$, of a sphere is given by $V = \frac{4}{3}\pi r^3$ and the surface area, $S \text{ cm}^2$, of a sphere is given by $S = 4\pi r^2$.

a Show that $V = \frac{1}{6\sqrt{\pi}} S^{\frac{3}{2}}$ (2 marks)

The surface area of the sphere is expanding at a constant rate of $4 \text{ cm}^2 \text{ s}^{-1}$.

b Show that $\frac{dV}{dt} = \sqrt{\frac{S}{\pi}}$ (2 marks)

Problem solving Set A

Bronze

The curve C has equation $y = x^4 - \frac{20}{3}x^3 + 14x^2 + 6x - 4$.

- a Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ (3 marks)
- b Show that the curve C is concave in the interval $\left[1, \frac{7}{3}\right]$. (3 marks)

Silver

The curve C has equation $y = 4x^3 \ln x^2$, $x > 0$.

Show that the curve C is convex when $x \geq e^{\frac{5}{6}}$. (6 marks)

Gold

$f(x) = ax^4 + 2x^3 + 3x^2 - 10x + 5$. Given that $f(x)$ is convex for all real values of x , find the range of possible values of a . (6 marks)

Problem solving Set B

Bronze

The curve C is described by $x^2 + y = (2 - x)(y - 1)$.

- a Find $\frac{dy}{dx}$ (4 marks)
- b Find the equation of the normal to C at the point with x -coordinate 0. (4 marks)

Silver

Circle C has parametric equations

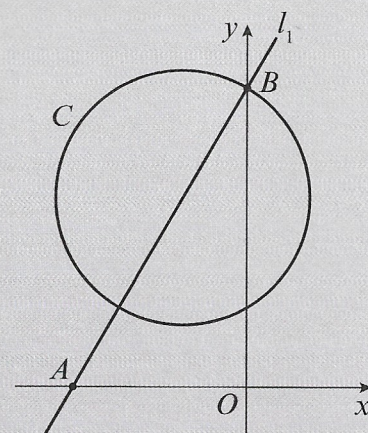
$$x = 2 \sin t - 1, y = 2 \cos t + 3, 0 \leq t \leq 2\pi$$

The line l_1 is normal to C at the point B , which lies on the y -axis.

- a Find the Cartesian equation of the circle, clearly stating the centre and radius. (3 marks)
- b Show clearly that B has coordinates $(0, 3 + \sqrt{3})$. (2 marks)

The line l_1 intersects the x -axis at the point A .

- c Find the area of triangle AOB . (7 marks)



Gold

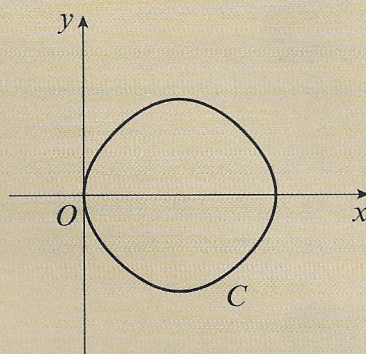
The diagram shows the curve C given by the equation

$$\cos y + \sin x = 1, \quad 0 \leq x \leq \pi, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Find the equations of the tangents to the curve that are parallel to the line $y = x$. You must show clear algebraic working.

Solutions based on graphical or numerical approaches are not acceptable.

(9 marks)

**Now try this**

→ Exam question bank Q4, Q7, Q28, Q39, Q43, Q52, Q67, Q77