

# 10 Trigonometric identities and equations

## 10.1 Angles in all four quadrants

- 1 Use the unit circle to determine the quadrants where the following trigonometric functions are positive.

- a  $\sin \theta$
- b  $\cos \theta$
- c  $\tan \theta$
- d all of  $\sin \theta$ ,  $\cos \theta$  and  $\tan \theta$

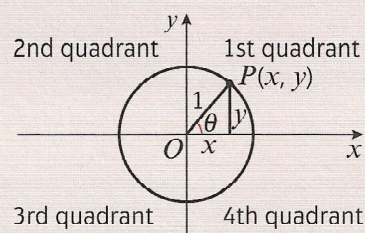
**Hint**  $P(x, y)$  moves on a unit circle such that  $OP$  makes an angle  $\theta$  with the positive  $x$  axis.

$$\cos \theta = x = x\text{-coordinate of } P$$

$$\sin \theta = y = y\text{-coordinate of } P$$

$$\tan \theta = \frac{y}{x} = \text{gradient of } OP$$

This means that the sign, (+ or -), of the trigonometric functions depends on the sign of  $x$  and  $y$ .



- 2 Identify the quadrant that  $\theta$  lies in as  $OP$  moves around the unit circle and hence state whether the trigonometric function of  $\theta$  is positive or negative.

- a  $\theta = 116^\circ$ ,  $\cos \theta$
- b  $\theta = 240^\circ$ ,  $\tan \theta$
- c  $\theta = 420^\circ$ ,  $\sin \theta$
- d  $\theta = -100^\circ$ ,  $\tan \theta$
- e  $\theta = 350^\circ$ ,  $\sin \theta$
- f  $\theta = -80^\circ$ ,  $\cos \theta$

- 3 Without using a calculator, write down the values of:

- a  $\cos 90^\circ$
- b  $\cos(-180^\circ)$
- c  $\sin 540^\circ$
- d  $\sin(-90^\circ)$
- e  $\tan 135^\circ$
- f  $\tan(-45^\circ)$
- g  $\sin 225^\circ$
- h  $\cos(-45^\circ)$
- i  $\tan(-90^\circ)$

**Hint** When  $OP$  makes an angle of  $45^\circ$  with the positive  $x$ -axis the triangle formed is isosceles.

$$\cos 45^\circ = x = \frac{1}{\sqrt{2}}$$

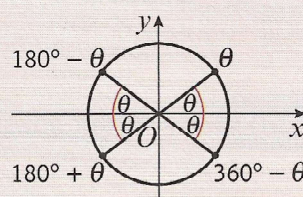
$$\sin 45^\circ = y = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{y}{x} = 1$$

- 4 Given that  $\theta$  is an acute angle, express in terms of  $\sin \theta$ ,  $\cos \theta$  or  $\tan \theta$ :

- a  $\sin(180^\circ + \theta)$
- b  $\cos(360^\circ - \theta)$
- c  $\tan(180^\circ - \theta)$
- d  $\sin(-\theta)$
- e  $\tan(180^\circ + \theta)$
- f  $\cos(\theta - 180^\circ)$
- g  $\sin(\theta - 360^\circ)$
- h  $\cos(540^\circ - \theta)$
- i  $\tan(\theta - 180^\circ)$

**Hint**



- P 5** Express in terms of trigonometric ratios of acute angles:

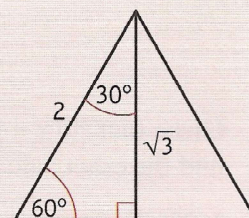
a  $\sin 220^\circ$       b  $\cos 160^\circ$       c  $\tan 210^\circ$   
 d  $\sin(-100^\circ)$       e  $\tan 130^\circ$       f  $\cos 280^\circ$   
 g  $\sin 95^\circ$       h  $\cos(-200^\circ)$       i  $\tan(-115^\circ)$

## 10.2 Exact values of trigonometric ratios

- 1** Express the following as trigonometric ratios of either  $30^\circ$ ,  $45^\circ$  or  $60^\circ$  and hence state the exact value.

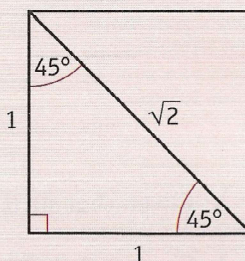
a  $\sin 150^\circ$       b  $\sin(-45^\circ)$   
 c  $\sin 315^\circ$       d  $\sin 240^\circ$   
 e  $\cos 150^\circ$       f  $\cos(-60^\circ)$   
 g  $\cos(-135^\circ)$       h  $\cos 120^\circ$   
 i  $\tan 225^\circ$       j  $\tan(-150^\circ)$   
 k  $\tan 405^\circ$       l  $\tan(-120^\circ)$

**Hint** The exact values of the trigonometric ratios of  $30^\circ$  and  $60^\circ$  can be worked out using Pythagoras' theorem by considering an equilateral triangle of side 2 units.



$$\begin{aligned} \sin 30^\circ &= \frac{1}{2} & \cos 30^\circ &= \frac{\sqrt{3}}{2} & \tan 30^\circ &= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\ \sin 60^\circ &= \frac{\sqrt{3}}{2} & \cos 60^\circ &= \frac{1}{2} & \tan 60^\circ &= \sqrt{3} \end{aligned}$$

The exact values of the trigonometric ratios of  $45^\circ$  can be worked out by considering a square of side 1 unit.



$$\begin{aligned} \sin 45^\circ &= \frac{1}{\sqrt{2}} \\ \cos 45^\circ &= \frac{1}{\sqrt{2}} & \tan 45^\circ &= 1 \end{aligned}$$

- 2** The table shows some of the values of  $x$  and  $y$  that satisfy the equation  $y = a \sin x + b$ .

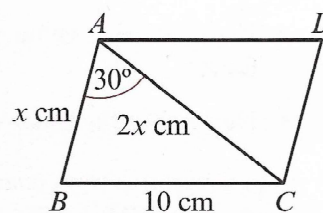
| $x$ | $0^\circ$ | $30^\circ$ | $60^\circ$     | $90^\circ$ | $120^\circ$    | $150^\circ$ | $180^\circ$ |
|-----|-----------|------------|----------------|------------|----------------|-------------|-------------|
| $y$ | 3         | 4          | $3 + \sqrt{3}$ | 5          | $3 + \sqrt{3}$ | 4           | 3           |

Find the value of  $y$  when  $x = 45^\circ$ .

- 3** The diagram shows parallelogram  $ABCD$ , where  $AB = DC = x$  cm and  $AD = BC = 10$  cm.

Given that  $AC = 2x$  cm and  $\angle BAC = 30^\circ$ , show that

$$x^2 = \frac{100(5 + 2\sqrt{3})}{13}$$



## 10.3 Trigonometric identities

- 1** Write down the value of:

a  $2 \sin^2 \theta - (1 - \cos^2 \theta) + \cos^2 \theta$   
 b  $6 \sin^2 \frac{\theta}{2} + 6 \cos^2 \frac{\theta}{2}$       c  $(1 + \cos x)^2 + (1 - \cos x)^2 + 2 \sin^2 x$

**Hint** Use the trigonometric identity  $\sin^2 \theta + \cos^2 \theta = 1$ .

2 Write in terms of  $\tan x$ :

a  $\frac{2 \sin x}{\cos x}$       b  $\frac{3 \sin x}{\sqrt{1 - \sin^2 x}}$       c  $\frac{1 - \cos^2 x}{1 - \sin^2 x}$

3 Prove that  $\frac{\sin^2 x - \cos^2 x}{1 - \sin^2 x} \equiv \tan^2 x - 1$

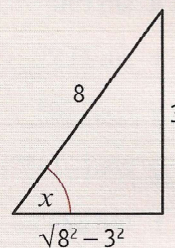
**Hint** Use the trigonometric identity  $\tan x = \frac{\sin x}{\cos x}$  for all values of  $x$  such that  $\cos x \neq 0$

**Hint** To prove an identity, start from the LHS and manipulate the expression until it matches the RHS. You can quote the standard identities in your proof. ← Sections 7.4, 7.5

4 Given that  $\sin x = \frac{3}{8}$  and that  $x$  is obtuse, find the exact value of:

a  $\cos x$       b  $\tan x$

**Hint** You can use  $\cos^2 x \equiv 1 - \sin^2 x$  to work out the exact value of  $\cos x$ , then  $\tan x = \frac{\sin x}{\cos x}$ . Remember you need to check the sign of cosine and tangent in the required quadrant. You can check your answer using basic trigonometry and Pythagoras' theorem in a right-angled triangle.



**E** 5 a Given that  $\sin \theta = 6 \cos \theta$ , find the value of  $\tan \theta$ . (1 mark)

b Express  $4 \cos^2 \theta - 5 \sin^2 \theta$  in terms of  $\sin \theta$ . (1 mark)

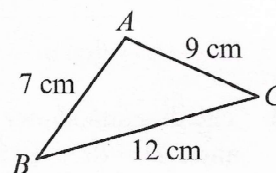
**E/P** 6 Show that the equation  $4 \cos^2 x - 3 \sin^2 x = 2$  can be written as  $7 \cos^2 x = 5$ . (2 marks)

**E/P** 7 Prove that  $\frac{\cos^2 x - \sin^2 x}{\cos^2 x - \sin x \cos x} \equiv 1 + \tan x$  (3 marks)

**E/P** 8 The diagram shows the triangle  $ABC$  with  $AB = 7$  cm,  $BC = 12$  cm and  $AC = 9$  cm.

a Find the exact value of  $\cos B$ , giving your answer in its simplest form. (3 marks)

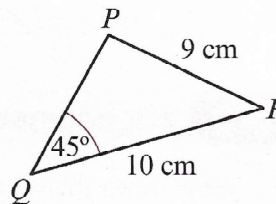
b Hence find the exact value of  $\sin B$ . (2 marks)



**E/P** 9 The diagram shows triangle  $PQR$  with  $PR = 9$  cm,  $QR = 10$  cm and angle  $PQR = 45^\circ$ .

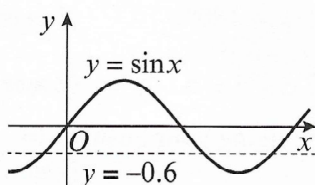
a Show that  $\sin P = \frac{5\sqrt{2}}{9}$  (3 marks)

b Given that angle  $P$  is obtuse, find the exact value of  $\cos P$ . (2 marks)



## 10.4 Simple trigonometric equations

- 1 The diagram shows a sketch of  $y = \sin x$ .



- a Use your calculator to find the principal solution to the equation  $\sin x = -0.6$ .
- b Use the graph and your answer to part a to find solutions to the equation  $\sin x = -0.6$  in the interval  $0 \leq x \leq 360^\circ$ .

**Hint** Use the inverse trigonometric function on your calculator to get the **principal value**.

Your calculator will give the principal values in these ranges:

$$\cos^{-1}: 0 \leq x \leq 180^\circ$$

$$\sin^{-1}: -90^\circ \leq x \leq 90^\circ$$

$$\tan^{-1}: -90^\circ < x < 90^\circ$$

- 2 Find the solutions to the equation

$$\cos \theta = \frac{1}{\sqrt{2}} \text{ in the interval } 0 \leq \theta \leq 360^\circ.$$

**Hint** Find the first solution using your calculator or your knowledge of exact values of trigonometric functions. Draw the graph of  $y = \cos \theta$  for the given interval and use the properties of the graph to find the second solution.

← Section 9.5

- 3 Solve the equation  $4 \sin x = 3$  in the interval  $0 \leq x \leq 360^\circ$ .

**Hint** First rewrite the equation in the form  $\sin x = \dots$

- 4 Solve the equation  $\sin \theta = -10 \cos \theta$  in the interval  $0 < \theta \leq 360^\circ$ .

**Hint** You can use the identity  $\tan \theta = \frac{\sin \theta}{\cos \theta}$  to solve trigonometric equations. The principal value of  $\tan \theta$  is not in the given interval. As the period for the graph of  $y = \tan \theta$  is  $180^\circ$ , you can find successive solutions by adding or subtracting multiples of  $180^\circ$ .

- E** 5 A student's attempt to solve the equation  $5 \tan x = 9 \sin x$  in the interval  $-90^\circ < x < 90^\circ$  is shown.

- a Identify two mistakes made by this student, giving a brief explanation of each mistake. (2 marks)
- b Write down the correct solutions to the original equation. (1 mark)

$$5 \tan x = 9 \sin x$$

$$5 \frac{\sin x}{\cos x} = 9 \sin x$$

$$5 \sin x = 9 \sin x \cos x$$

$$5 = 9 \cos x$$

$$\cos x = \frac{5}{9}$$

$$x = 56.3^\circ \text{ (to 3 s.f.)}$$