

1. Using the substitution $u = \cos x + 1$, or otherwise, show that

$$\int_0^{\pi/2} e^{\cos x + 1} \sin x \, dx = e - 1$$

(Total 6 marks)

$$u = \cos x + 1 \quad \frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{1}{\sin x} du$$

$$x=0 \quad u = \cos(0) + 1 = 2$$

$$x=\pi/2 \quad u = \cos(\pi/2) + 1 = 1$$

$$\int_2^1 e^u \sin x \left(-\frac{1}{\sin x}\right) du$$

$$= [-e^u]_2^1 = -e^1 - (-e^2)$$

$$= e^2 - e^1$$

$$= e(e-1)$$

2. (a) Using the substitution $x = 2 \cos u$, or otherwise, find the exact value of

$$\int_1^{\sqrt{2}} \frac{1}{x^2 \sqrt{4-x^2}} dx$$

(7)

$$\frac{dx}{du} = -2 \sin u \quad dx = -2 \sin u \, du$$

$$x=1 \therefore 1 = 2 \cos u \Rightarrow \cos u = \frac{1}{2} \Rightarrow u = \pi/6$$

$$x=\sqrt{2} \quad \sqrt{2} = 2 \cos u \Rightarrow \cos u = \frac{\sqrt{2}}{2} \Rightarrow u = \pi/4$$

$$\int_{\pi/6}^{\pi/4} \frac{1}{(2 \cos u)^2 \sqrt{4 - (2 \cos u)^2}} (-2 \sin u) du$$

$$= \int_{\pi/6}^{\pi/4} \frac{1}{4 \cos^2 u \sqrt{4(1 - \cos^2 u)}} (-2 \sin u) du$$

$$= \int_{\pi/6}^{\pi/4} \frac{1}{4 \cos^2 u \cancel{2 \sin u}} \cancel{(-2 \sin u)} du$$

$$= \int_{\pi/6}^{\pi/4} \frac{-1}{4 \cos^2 u} du$$

$$= -\frac{1}{4} \int_{\pi/6}^{\pi/4} \sec^2 u \, du = -\frac{1}{4} [\tan u]_{\pi/6}^{\pi/4}$$

3. (a) Find $\int \tan^2 x \, dx$.

$$\int \sec^2 x \, dx = \tan x + C$$

$$\sec^2 x = 1 + \tan^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

$$\int (\sec^2 x - 1) \, dx = \tan x - x + C$$

Rules

1) When you see $\int \sin^2 x \, dx$ or $\int \cos^2 x \, dx$
you think $\cos 2x = 2\cos^2 x - 1$
 $= 1 - 2\sin^2 x$

2) $\int \tan^2 x \, dx$ then $\sec^2 x = 1 + \tan^2 x$ a) $\int \frac{1 - \cos 2x}{2} \, dx$ b) $\int \left(\frac{\cos 2x - 1}{2} \right) \, dx$

b) $\int \ln x \, dx = \int 1 \ln x \, dx$

$$u = \ln x \quad \frac{du}{dx} = \frac{1}{x} \quad = uv - \int v \frac{du}{dx} \, dx$$

$$\frac{dv}{dx} = 1 \quad v = x \quad = x \ln x - \int \frac{1}{x} \times x \, dx$$

$$= x \ln x - x + C$$

(c) Use the substitution $u = 1 + e^x$ to show that

$$\int \frac{e^{3x}}{1 + e^x} \, dx = \frac{1}{2} e^{2x} - e^x + \ln(1 + e^x) + k,$$

where k is a constant.

$$\frac{du}{dx} = e^x \Rightarrow dx = \frac{1}{e^x} du$$

$$\int \frac{e^{3x}}{u} \cdot \frac{1}{e^x} du = \int \frac{e^{2x}}{u} du$$

$$= \int \frac{(u-1)^2}{u} du$$

$$= \int \frac{u^2 - 2u + 1}{u} du$$

$$= \int \left(u - 2 + \frac{1}{u} \right) du$$

$$= \frac{u^2}{2} - 2u + \ln u + k$$

$$e^x = u - 1$$

$$e^{2x} = (u-1)^2$$

(7)

4. Use the substitution $u = 2^x$ to find the exact value of

$$\int_0^1 \frac{2^x}{(2^x + 1)^2} dx.$$

(Total 6 marks)

$$\frac{du}{dx} = 2^x \ln 2$$

$$= u \ln 2$$

$$\int \frac{\cancel{u}}{(u+1)^2} \cdot \frac{1}{\cancel{u \ln 2}} du$$

$$dx = \frac{1}{u \ln 2} du$$

$$= \frac{1}{\ln 2} \int \frac{1}{(u+1)^2} du$$

$$= \frac{1}{\ln 2} \int (u+1)^{-2} du$$

$$= \frac{1}{\ln 2} \left(-1 \frac{(u+1)^{-1}}{1} \right) + C$$

$$= -\frac{1}{(u+1) \ln 2}$$

6. Use the substitution $x = \sin \theta$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}} dx.$$

(Total 7 marks)

$$\frac{dx}{d\theta} = \cos \theta$$

$$dx = \cos \theta d\theta$$

$$x=0 \quad 0 = \sin \theta \quad \theta=0$$

$$x=1/2 \quad 1/2 = \sin \theta \quad \theta = \pi/6$$

$$\int_0^{\pi/6} \frac{1}{\sqrt{(1-\sin^2 \theta)}^3} \cos \theta d\theta$$

$$= \int_0^{\pi/6} \frac{1}{\cos^3 \theta} \cos \theta d\theta$$

$$= \int_0^{\pi/6} \sec^2 \theta d\theta$$

$$= [\tan \theta]_0^{\pi/6}$$

7. Use the substitution $u = 1 + \sin x$ and integration to show that

$$\int \sin x \cos x (1 + \sin x)^5 dx = \frac{1}{42} (1 + \sin x)^6 [6 \sin x - 1] + \text{constant}.$$

(Total 8 marks)

$$\frac{du}{dx} = \cos x \Rightarrow dx = \frac{1}{\cos x} du$$

$$\int (u-1) \cancel{\cos x} u^5 \frac{1}{\cancel{\cos x}} du = \int u^6 - u^5 du$$

8. Use the substitution $u^2 = (x-1)$ to find

$$\int \frac{x^2}{\sqrt{(x-1)}} dx,$$

$$x = u^2 + 1$$

giving your answer in terms of x .

(Total 10 marks)

$$\text{implicit } 2u \frac{du}{dx} = 1 \Rightarrow dx = 2u du$$

$$\int \frac{(u^2+1)^2}{\sqrt{u^2}} \cancel{2u} du$$

$$\int 2(u^4 + 2u^2 + 1) du$$

$$* \text{ i) } 6 \sin x + 5 \cos x = 0$$

$$6 \sin x = -5 \cos x \Rightarrow \tan x = -\frac{5}{6} \quad \checkmark$$

$$\text{ii) } 6 \sin x + 5 \cos x = 5$$

$$R \cos(x-\alpha)$$

$$R \cos x \cos \alpha + R \sin x \sin \alpha = 6 \sin x + 5 \cos x$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{6}{5}$$

$$\tan \alpha = \frac{6}{5} \Rightarrow \alpha = 50.2^\circ \text{ (s.s.f.)}$$

$$R^2 = 6^2 + 5^2$$

$$R = \sqrt{61}$$

$$\sqrt{61} \cos(x - 50.2) = 5$$

$$x - 50.2 = \cos^{-1}\left(\frac{5}{\sqrt{61}}\right)$$

a Express $\frac{15}{2} \sin 4x - 4 \cos 4x - 4 \cos 2x - 4$ in the form $\cos 2x(a \sin 2x - b \cos 2x - c)$, where a , b and c are constants to be found. (3 marks)

Hence or otherwise:

b solve $\frac{15}{2} \sin 4x = 4(\cos 4x + \cos 2x + 1)$ in the interval $0 \leq x \leq \pi$, giving your answers to 2 decimal places (4 marks)

c deduce the maximum value of $\frac{1}{30 \sin x \cos x - 16 \cos^2 x}$ (4 marks)

$$a) \frac{15}{2} (2 \sin 2x \cos 2x) - 4(2 \cos^2 2x - 1) - 4 \cos 2x - 4$$

$$15 \sin 2x \cos 2x - 4 \cos^2 2x + 4 - 4 \cos 2x - 4$$

$$= \cos 2x (15 \sin 2x - 4 \cos 2x - 4)$$

$$b) \cos 2x (15 \sin 2x - 4 \cos 2x - 4) = 0$$

$$\cos 2x = 0 \Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$15 \sin 2x - 4 \cos 2x - 4 = 0$$

$$R \sin(2x - \alpha) \text{ or } R \cos(2x + \alpha)$$

↓

$$1 \sin 2x \cos \alpha - 1 \cos 2x \sin \alpha = 15 \sin 2x - 4 \cos 2x$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{4}{15}$$

$$\tan \alpha = \frac{4}{15} \Rightarrow \alpha = 0.26 \text{ rad (2 d.p.)}$$

$$R = \sqrt{4^2 + 15^2}$$

$$R = \sqrt{241}$$

$$\sqrt{241} \sin(2x - 0.26) - 4 = 0$$

$$\sin(2x - 0.26) = \frac{4}{\sqrt{241}} \Rightarrow \dots$$

c deduce the maximum value of $\frac{1}{30 \sin x \cos x - 16 \cos^2 x}$

$$30 \sin x \cos x - 16 \cos^2 x = 15 \sin 2x - 16 \cos^2 x$$

$$\cos 2x = 2 \cos^2 x - 1$$

$$\cos^2 x = \frac{\cos 2x + 1}{2}$$

$$= 15 \sin 2x - 16 \left(\frac{\cos 2x + 1}{2} \right)$$

$$= 15 \sin 2x - 8 \cos 2x - 8$$

$$= 2 \left(\frac{15}{2} \sin 2x - 4 \cos 2x - 4 \right)$$

$$= 2 \left(\sqrt{241} \sin(2x - 0.26) \right)$$

$$\frac{1}{2(241 \sin(2x - 0.18) - 4)}$$

max when denominator smallest

$$\frac{1}{2(0 - 4)} =$$