

6 Circles

6.1 Midpoints and perpendicular bisectors

- 1 Find the midpoint of the line segment joining each pair of points:

- a $(2, -5), (-4, 7)$
- b $(-3, 9), (-1, -5)$
- c $\left(\frac{9}{5}, -\frac{3}{8}\right), \left(-\frac{3}{5}, -\frac{1}{8}\right)$
- d $(2\sqrt{3}, -\sqrt{5}), (-\sqrt{3}, -3\sqrt{5})$
- e $(1 + 2\sqrt{2}, 3 - \sqrt{2}), (3 + 2\sqrt{2}, 1 - 3\sqrt{2})$

Hint The **midpoint** of the line segment with endpoints (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

- 2 Find the equation of the perpendicular bisector of the line segment joining each pair of points. Give your answer in the form $ax + by + c = 0$.

- a $(-3, 4), (1, -8)$
- b $(3, -5), (-7, 15)$
- c $(2, -3), (-6, -9)$

Hint The **perpendicular bisector** of a line segment AB is the straight line that is perpendicular to AB and passes through the midpoint of AB . To find its equation:

- find the coordinates of the midpoint (x_1, y_1)
- find the gradient, M , of the line segment AB
← Section 5.1
- find the gradient, m , of the perpendicular to AB using $m = -\frac{1}{M}$
← Section 5.3
- use $y - y_1 = m(x - x_1)$
← Section 5.2

- 3 The line segment AB is a diameter of a circle, where A is $(-3, 4)$ and B is $(5, 8)$. Find:

- a the coordinates of the centre of the circle
- b the radius of the circle in the form $k\sqrt{5}$ where k is a constant to be found.

Hint For part **b**, first find the diameter, d , which is the length of the line segment joining the points (x_1, y_1) , (x_2, y_2) , and is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
← Section 5.4

- 4 The line segment PQ has endpoints $P(3, -5)$ and $Q(a, b)$. The midpoint of PQ is $M(6, -3)$. Find the values of a and b .

- 5 Points P , Q and R have coordinates $(-1, -3)$, $(5, 7)$ and $(7, -5)$ respectively. Find:

- a the equation of the perpendicular bisector of PQ
- b the equation of the perpendicular bisector of QR
- c the exact coordinates of the point of intersection of the two perpendicular bisectors.

- E/P** 6 The line segment PQ , where P is $(3, 8)$ and Q is $(-1, -4)$, is the diameter of a circle with centre C .
The line l is perpendicular to PQ and passes through C . Find an equation of l . (7 marks)

- E/P 7** The line segment PQ is a diameter of a circle, where P is $(-3, 6)$ and Q is $(5, -2)$. Find:
- the coordinates of the centre of the circle (2 marks)
 - the radius of the circle in the form $k\sqrt{2}$, where k is a constant to be found. (3 marks)
- R and S are the points where the perpendicular bisector of PQ meets the circle.
- Show that $PR = RQ = QS = SP = 8$. (4 marks)
- E/P 8** The points A and B have coordinates $(2k - 1, -3)$ and $(3, 3k + 7)$ respectively, where k is a constant. The coordinates of the midpoint of AB are $(5, p)$, where p is a constant.
- Find the value of k . (2 marks)
 - Find the value of p . (2 marks)
 - Find an equation of the perpendicular bisector of AB , giving your answer in the form $ax + by + c = 0$, where a , b and c are integers. (6 marks)

6.2 Equation of a circle

- 1 Write down the equation of each circle:

- Centre $(-5, 3)$, radius 8
- Centre $(7, -8)$, radius 10
- Centre $(-1, -4)$, radius $3\sqrt{2}$

Hint The equation of a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$

- 2 Write down the coordinates of the centre and the radius of each circle:

- $(x + 2)^2 + (y - 9)^2 = 144$
- $(x - 5)^2 + (y + 2)^2 = 32$
- $(x + 6)^2 + y^2 = 45$

- 3 Find the centre and the radius of the circle with the equation $x^2 + y^2 + 8x - 12y - 15 = 0$

Hint To find the coordinates of the centre and the radius, first complete the square for the terms in x and for the terms in y separately. ← Section 2.2

- 4 The line segment PQ is a diameter of a circle, where P is $(-1, 8)$ and Q is $(7, 20)$. Find the equation of the circle.

Hint Find the coordinates of the centre by finding the midpoint of PQ using $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. Find the radius of the circle by finding the diameter and dividing by 2.

- E/P 5** The circle with equation $(x - 1)^2 + (y - k)^2 = 50$ passes through the point $(2, 3)$. Find the possible values of k . (5 marks)

- E/P 6** A circle has equation $x^2 + y^2 - 6x + 10y - 16 = 0$. Find:
- the coordinates of the centre of the circle (3 marks)
 - the radius of the circle in the form $k\sqrt{2}$, where k is an integer to be found. (3 marks)

- E** 7 The circle C has centre $(1, 6)$ and passes through the point $P(-3, 4)$.
 a Find an equation for C . (4 marks)

C crosses the y -axis at the points A and B .

- b Find the coordinates of the points A and B . (3 marks)

- E/P** 8 The midpoint of the line joining the points $P(5, 8)$ and $Q(p, q)$ is $M(-2, 3)$.

- a Find the values of p and q . (3 marks)

PQ is the diameter of a circle.

- b Find the radius of the circle. (2 marks)

- c Find the equation of the circle, giving your answer in the form $x^2 + y^2 + ax + by + c = 0$. (3 marks)

- E/P** 9 The circle with equation $x^2 + y^2 + 4y - k = 0$ has radius 5.

- a Find the coordinates of the centre of the circle and the value of k . (4 marks)

The points $P(p, 2)$ and $Q(-4, 1)$, where $p > 0$, lie on the circumference of the circle.

- b Calculate the length of PQ , giving your answer in simplified surd form. (5 marks)

6.3 Intersections of straight lines and circles

- 1 Find the coordinates of the points where the circle $(x - 2)^2 + (y - 4)^2 = 52$ meets:

- a the x -axis b the y -axis

Hint The circle meets the x -axis when $y = 0$ and meets the y -axis when $x = 0$.

- 2 The line with equation $2x + y = 7$ meets the circle $(x - 1)^2 + y^2 = 50$ at points P and Q .

Find the coordinates of P and Q .

Hint A straight line may not intersect a circle, or can intersect once (by just touching the circle), or can intersect twice. To find the points of intersection, solve simultaneously by substituting the linear equation into the equation for the circle and solving the resulting quadratic equation.

← Section 3.2

- 3 a Show that the line $2x + y - 15 = 0$ meets the circle with equation $(x + 2)^2 + (y - 4)^2 = 45$ at exactly one point.

- b Find the coordinates of the point of intersection.

Hint To find the number of points of intersection you can use the discriminant for the resulting quadratic equation $ax^2 + bx + c = 0$ where:

- $b^2 - 4ac > 0$ indicates 2 points of intersection – the line cuts the circle twice
- $b^2 - 4ac = 0$ indicates 1 point of intersection – the line is a tangent to the circle
- $b^2 - 4ac < 0$ indicates 0 points of intersection – the line does not meet the circle.

← Section 2.5

- 4 Show that the line $y = 2x + 6$ does not meet the circle $(x - 5)^2 + (y - 2)^2 = 30$.
- E** 5 a Describe completely the curve with equation $x^2 + y^2 = 100$. (2 marks)
 b Find the coordinates of the points of intersection of the curve $x^2 + y^2 = 100$ and the line $3x + y - 10 = 0$. (6 marks)
- E/P** 6 A circle has equation $x^2 + y^2 - 6x - 23 = 0$.
 A line has equation $y = x + k$, where k is a constant.
 a Show that the x -coordinate of any point of intersection of the line and the circle satisfies the equation $2x^2 + 2(k - 3)x + k^2 - 23 = 0$. (3 marks)
 b Find the values of k for which the equation $2x^2 + 2(k - 3)x + k^2 - 23 = 0$ has equal roots. (4 marks)
 c Describe the geometrical relationship between the line and the circle when k takes either of the values found in part b. (1 mark)
- E/P** 7 The circle with equation $(x - 3)^2 + (y - 5)^2 = 61$ meets the straight line with equation $x + 11y = 119$ at points P and Q .
 a Find the coordinates of points P and Q . (5 marks)
 b Find an equation of the perpendicular bisector of line segment PQ . (5 marks)
 c Show that the perpendicular bisector passes through the centre of the circle. (2 marks)
- E/P** 8 The circle C has equation $x^2 + y^2 - 10x + 8y + 25 = 0$. The line with equation $y = kx$, where k is a constant, cuts C at two distinct points. Find the range of possible values for k . (6 marks)
- E/P** 9 The line $x + 3y = k$ meets the circle with equation $(x - p)^2 + (y - 4)^2 = 45$ at the point $(4, 7)$.
 a Work out the value of k . (1 mark)
 b Work out the two possible values of p . (5 marks)

6.4 Use tangent and chord properties

- 1 The circle C has equation $x^2 + y^2 - 6x + 4y - 156 = 0$.

- a Verify that the point $P(-2, 10)$ lies on C .
 b Find an equation of the tangent to C at the point P , giving your answer in the form $ax + by + c = 0$.

Hint

A tangent to a circle is perpendicular to the radius of the circle at the point of intersection.

For part b, first find the gradient, M , of the radius.

The gradient of the tangent is $m = -\frac{1}{M}$

The equation of the tangent at $P(x_1, y_1)$ can then be found using $y - y_1 = m(x - x_1)$.

- 2 The point $P(4, -6)$ lies on the circle with centre $(8, 2)$.

- a Find the equation of the circle.
b Find an equation of the tangent to the circle at P .

Hint The radius of the circle is the distance between P and the centre.

- 3 The equation of a circle is $x^2 + y^2 + 6x - 8y - 16 = 0$. The points $A(2, 8)$ and $B(-7, 9)$ lie on the circle.

- a Find an equation of the perpendicular bisector of the chord AB , giving your answer in the form $y = mx + c$.
b Show that the perpendicular bisector of AB passes through the centre of the circle.

Hint The perpendicular bisector of a chord of a circle always passes through the centre of the circle. Start by finding both the midpoint and gradient of the chord AB .

- 4 A circle has equation $(x - p)^2 + (y - 3)^2 = 20$, where $p \neq 0$ is a constant. The points $P(-2, 7)$ and $Q(-8, k)$ lie on the circle.

- a Find the value of p and the two possible values of k .
b For the greater possible value of k , find the equation of the perpendicular bisector of PQ , and show that it passes through the centre of the circle.

- E** 5 The line joining the points $A(4, p)$ and $B(q, -2)$ has midpoint $M(1, -4)$.

- a Find the values of p and q . (3 marks)

AB is the diameter of a circle.

- b Find the radius of the circle. (2 marks)

- c Find the equation of the circle in the form $x^2 + y^2 + ax + by + c = 0$. (3 marks)

- d Find an equation of the tangent to the circle at the point $(3, -9)$, giving your answer in the form $ax + by + c = 0$. (5 marks)

- E/P** 6 A circle C with centre $(-5, 9)$ passes through the point $(8, 14)$.

- a Show that the point $(8, 4)$ also lies on C . (3 marks)

The tangent to C at the point $(8, 14)$ meets the y -axis at the point A and the tangent to C at the point $(8, 4)$ meets the y -axis at the point B .

- b Work out the distance AB . (7 marks)

- E/P** 7 The points $P(1, 10)$ and $Q(7, 8)$ lie on the circumference of a circle with centre $C(3, k)$.

- a Find the equation of the perpendicular bisector of PQ . (5 marks)

- b Find the value of k . (2 marks)

- c Find the equation of the circle. (3 marks)

- d Find the exact coordinates of the points of intersection of the perpendicular bisector of PQ , and the circle. (5 marks)

- 8** The points $P(-2, 3)$ and $Q(8, 5)$ lie on a circle with centre $(4, -1)$.

a Find the equation of the circle.

(3 marks)

The line l passes through the centre of the circle and the midpoint of the chord PQ .

b Find an equation of l .

(5 marks)

The line l crosses the y -axis at A and the x -axis at B .

c Find the area of triangle AOB , where O is the origin.

(4 marks)

- 9** The circle C has equation $x^2 + y^2 - 2x + 6y - 30 = 0$.
The points $P(3, 3)$ and $Q(7, -5)$ lie on C .

a Find an equation for the tangent to C at Q , giving your answer in the form $ax + by + c = 0$, where a , b and c are integers.

(5 marks)

b Find an equation of the perpendicular bisector of PQ .

(5 marks)

c Find the coordinates of the point of intersection of the tangent and the perpendicular bisector.

(4 marks)

6.5 Circles and triangles

- 1** The vertices of a triangle are $A(-1, 6)$, $B(3, 4)$ and $C(5, 2)$.

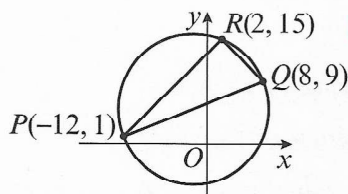
a Find the equations of the perpendicular bisectors of AB , BC and AC respectively.

b Show that the perpendicular bisectors all meet at the same point.

c Find the equation of the circle that passes through A , B and C .

Hint A unique circle, called the **circumcircle**, can be drawn through the three vertices of any triangle. The perpendicular bisectors of the three sides meet at the centre of the circle.

- 2** The points $P(-12, 1)$, $Q(8, 9)$ and $R(2, 15)$ lie on the circle as shown in the diagram.



a Show that PQ is a diameter of the circle.

b Find an equation of the circle in the form $(x - a)^2 + (y - b)^2 = c$, where a , b and c are constants to be found.

Hint Use Pythagoras' theorem to determine whether a triangle is right-angled.

The angle in a semicircle is 90° , so the hypotenuse of a right-angled triangle is a diameter of the circumcircle. The centre of the circle is at the midpoint of the hypotenuse.

- 3** A triangle ABC has vertices $A(6, 3)$, $B(-2, 1)$ and $C(0, -7)$.

a Show that ABC is an isosceles right-angled triangle.

b Find the equation of the circumcircle of triangle ABC .

- 4 The points $A(6, 14)$, $B(14, 10)$ and $C(-4, 4)$ lie on the circumference of a circle. The equation of the perpendicular bisector of AC is $x + y = 10$.

- Find the coordinates of the centre of the circle.
- Find the equation of the circle.

- E/P** 5 A circle with centre C has equation $x^2 + y^2 - 10x + 6y = 91$.

- Find the coordinates of C . (2 marks)

The points $A(0, 7)$ and $B(10, 7)$ lie on the circle.

- Find the area of triangle ABC . (4 marks)

- E/P** 6 The circle C has equation $x^2 + y^2 - 4x + 8y = 33$.

- Find the centre and radius of C . (5 marks)

The points $P(-5, -2)$ and $Q(9, -6)$ both lie on C .

- Show that PQ is a diameter of C . (2 marks)

The point R lies on the positive y -axis and the angle $PRQ = 90^\circ$.

- Find the coordinates of R . (4 marks)

- E/P** 7 A circle has equation $x^2 + y^2 + 4x - 6y - 23 = 0$.

- Find the coordinates of the centre of the circle. (2 marks)

- Show that the radius of the circle is 6. (2 marks)

The points A , B and C lie on the circle. The length of AB is 12 and the length of AC is 5.

- Find the length of BC , giving your answer to 1 decimal place. (3 marks)

- E/P** 8 The points P , Q and R have coordinates $(-2, 3)$, $(5, -1)$ and $(-3, k)$ respectively. Given that PQ is perpendicular to QR ,

- find the value of k . (4 marks)

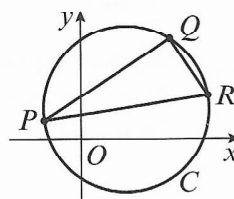
P , Q and R lie on the circumference of a circle.

- Find the coordinates of the centre of the circle. (3 marks)

- E/P** 9 The points $P(-4, 4)$, $Q(8, 12)$ and $R(k, 6)$ lie on the circle C as shown in the diagram.

Given that PR is a diameter of C ,

- find the value of k (3 marks)
- find an equation for C . (5 marks)



Problem solving Set A

Bronze

The circle C has centre $(0, 5)$ and radius 10.

- a Write down an equation of the circle. (3 marks)

The straight line with equation $2x + y = 1$ intersects C at the points P and Q .

- b Find the coordinates of P and Q . (6 marks)

Silver

Show that the line with equation $4y + 3x = 44$ is a tangent to the circle with centre $(5, 1)$ and radius 5. (7 marks)

Gold

The line with equation $5y - 4x = k$, where k is a constant, does not intersect the circle with equation $(x - 3)^2 + (y + 2)^2 = 41$. Find the range of possible values of k . (9 marks)

Problem solving Set B

Bronze

The diagram shows a sketch of the circle C with centre P .

The circle has equation $(x - 2)^2 + (y + 3)^2 = 25$.

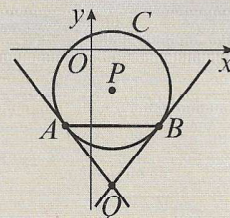
- a Write down the coordinates of P . (2 marks)
b State the radius of C . (1 mark)

The chord AB of C is parallel to the x -axis, lies below the x -axis and is of length 8.

- c Find the coordinates of points A and B . (5 marks)
d Write down the coordinates of M , the midpoint of AB . (1 mark)
e Show that angle $APM = 53.1^\circ$, to one decimal place. (2 marks)

The tangents to C at the points A and B meet at the point Q .

- f Find the length of AQ , giving your answer to 2 significant figures. (2 marks)



Silver

The diagram shows a sketch of the circle C with centre P .

The circle has equation $(x - 4)^2 + (y + 5)^2 = \frac{625}{4}$

a Write down the coordinates of P .

(2 marks)

b Find the radius of C .

(1 mark)

The chord AB of C is parallel to the x -axis, lies below the x -axis and has length 24.

c Find the coordinates of A and B .

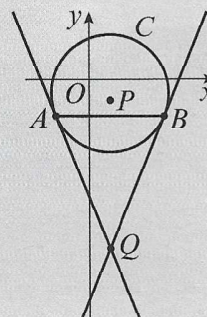
(5 marks)

d Show that angle $APB = 147.5^\circ$, to the nearest 0.1° .

(2 marks)

e Find the length AQ , giving your answer to 3 significant figures.

(2 marks)



Gold

The diagram shows a sketch of the circle C with centre P .

The circle has equation $(x - 3)^2 + (y + 2)^2 = \frac{289}{4}$

The chord AB of C is parallel to the x -axis, lies below the x -axis and has length 15.

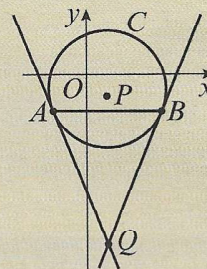
a Find the coordinates of A and B .

(8 marks)

The tangents to C at the points A and B meet at the point Q .

b Find the length of AQ , giving your answer to 3 significant figures.

(4 marks)



Now try this

→ Exam question bank Q28, Q42, Q58, Q70, Q82, Q87, Q92