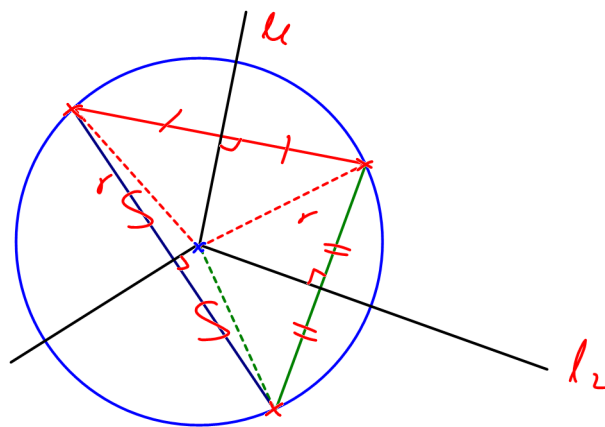
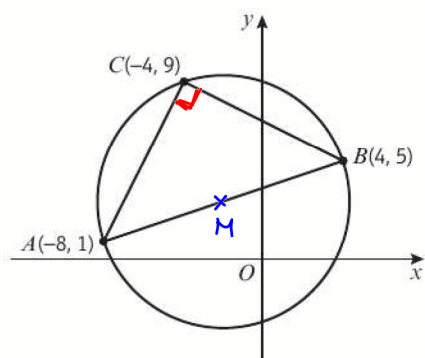




Example 13

The points $A(-8, 1)$, $B(4, 5)$ and $C(-4, 9)$ lie on the circle, as shown in the diagram.

- Show that AB is a diameter of the circle.
- Find an equation of the circle.



$$m_{AC} = \frac{9-1}{-4+8} = \frac{8}{4} = 2$$

$$m_{CB} = \frac{5-9}{4+4} = \frac{-4}{8} = -\frac{1}{2}$$

m_{AC} is the negative reciprocal of m_{CB}
 $m_{AC} \times m_{CB} = 2 \times (-\frac{1}{2}) = -1$ perpendicular

$$M\left(-\frac{8+4}{2}, \frac{1+5}{2}\right) \Rightarrow M(-2, 3)$$

$$(x-a)^2 + (y-b)^2 = r^2$$

$$r = \frac{|AB|}{2} \Rightarrow |AB| = \sqrt{(4+8)^2 + (5-1)^2} = 4\sqrt{10}$$

$$r = 2\sqrt{10}$$

$$(x+2)^2 + (y-3)^2 = (2\sqrt{10})^2$$

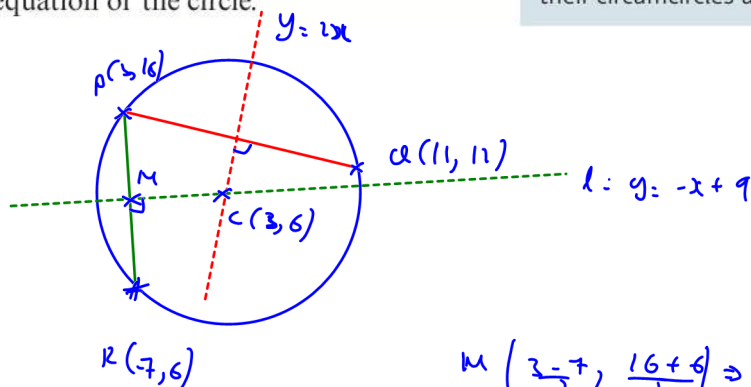
$$(x+2)^2 + (y-3)^2 = 40$$

Example 14

The points $P(3, 16)$, $Q(11, 12)$ and $R(-7, 6)$ lie on the circumference of a circle. The equation of the perpendicular bisector of PQ is $y = 2x$.

- Find the equation of the perpendicular bisector of PR .
- Find the centre of the circle.
- Work out the equation of the circle.

Online Explore triangles and their circumcircles using GeoGebra.



$$M\left(\frac{3-7}{2}, \frac{16+6}{2}\right) \Rightarrow M(-2, 11)$$

$$m_{PR} = \frac{6-16}{-7-3} = \frac{-10}{-10} = 1 \quad l: m = -1$$

$$y - y_1 = m(x - x_1)$$

$$y - 11 = -1(x + 2) \Rightarrow y = -x + 9$$

$$b) \quad l: y = 2x \quad l: y = -x + 9$$

$$2x = -x + 9 \Rightarrow 3x = 9 \Rightarrow x = 3$$

$$y = -x + 9 \quad y = 6$$

$$C(3, 6)$$

$$P(3, 16) \quad C(3, 6)$$

$$r = |PC| \Rightarrow |PC| = \sqrt{(3-3)^2 + (6-16)^2} = \sqrt{0+100} = 10$$

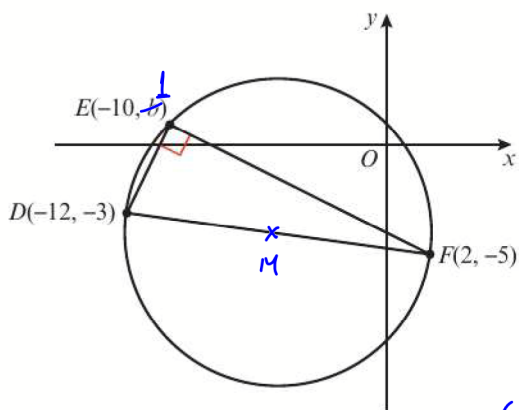
$$C(3, 6)$$

$$(x-3)^2 + (y-6)^2 = 100$$

- 8 The points $D(-12, -3)$, $E(-10, b)$ and $F(2, -5)$ lie on the circle C as shown in the diagram.

Given that $\angle DEF = 90^\circ$ and $b > 0$

- a show that $b = 1$ (5 marks)
b find an equation for C . (4 marks)



$$m_{ED} \times m_{EF} = -1$$

$$m_{ED} = \frac{-3-b}{-12+10} = \frac{-3-b}{-2} = \frac{3+b}{2}$$

$$m_{EF} = \frac{-5-b}{2+10} = \frac{-5-b}{12}$$

$$\left(\frac{3+b}{2}\right) \left(\frac{-5-b}{12}\right) = -1$$

$$\frac{(3+b)(-5-b)}{2 \times 12} = -1 \Rightarrow -15 - 8b - b^2 = -24$$

$$b^2 + 8b - 24 + 15 = 0 \quad b^2 + 8b - 9 = 0$$

$$(b+9)(b-1) = 0$$

$$b = -9 \quad \text{or} \quad \boxed{b = 1}$$

$$b > 0$$

$$D(-12, -3)$$

$$F(2, -5)$$

$$M\left(\frac{-12+2}{2}, \frac{-3-5}{2}\right) \Rightarrow M(-5, -4)$$

$$|MF| = r \quad |MF| = \sqrt{(2+5)^2 + (-5+4)^2}$$

$$= \sqrt{49 + 1}$$

$$r = \sqrt{50}$$

$$(x+5)^2 + (y+4)^2 = 50$$

- 9 A circle has equation $x^2 + 2x + y^2 - 24y - 24 = 0$

a Find the centre and radius of the circle. (3 marks)

b The points $A(-13, 17)$ and $B(11, 7)$ both lie on the circumference of the circle. Show that AB is a diameter of the circle. (3 marks)

c The point C lies on the negative x -axis and the angle $ACB = 90^\circ$.

Find the coordinates of C . (3 marks)

a)

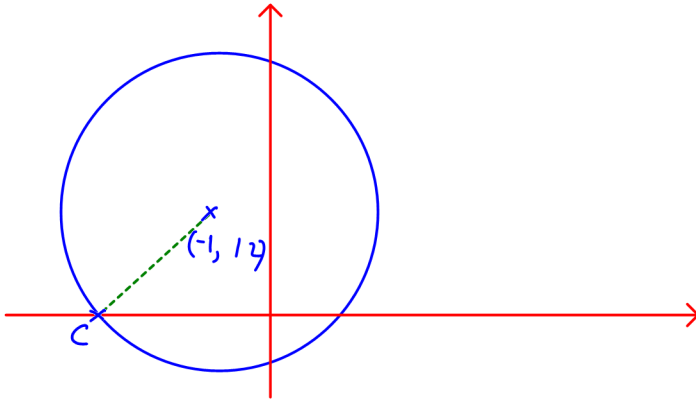
$$(x+1)^2 - 1^2 + (y-12)^2 - 12^2 - 24 = 0$$

$$(x+1)^2 + (y-12)^2 - 169 = 0 \quad (x+1)^2 + (y-12)^2 = 169$$

$$C(-1, 12) \quad r = 13$$

$A(-13, 17)$ and $B(11, 7)$

$$\left(\frac{-13+11}{2}, \frac{17+7}{2} \right) = (-1, 12)$$



$$(x+1)^2 + (y-12)^2 = 169 \quad \text{at } C \quad y=0$$

$$(x+1)^2 + (0-12)^2 = 169 \Rightarrow (x+1)^2 + 144 = 169$$

$$(x+1)^2 = 25$$

$$x+1 = \pm 5 \Rightarrow x = 5-1 = 4$$

$$x = -5-1 = -6$$

$$C(-6, 0)$$

Example 2

Divide $x^3 + 2x^2 - 17x + 6$ by $(x - 3)$.

$$\begin{array}{r} \swarrow x^2 + 5x - 2 \\ x-3 \overline{) x^3 + 2x^2 - 17x + 6} \\ \underline{-(x^3 - 3x^2)} \\ \phantom{x-3 \overline{) }} + 5x^2 - 17x \\ \phantom{x-3 \overline{) }} \underline{5x^2 - 15x} \\ \phantom{x-3 \overline{) }} - 2x + 6 \\ \phantom{x-3 \overline{) }} \underline{-2x + 6} \\ \phantom{x-3 \overline{) }} 0 \end{array}$$

$$x^3 + 2x^2 - 17x + 6 = (x-3)(x^2 + 5x - 2)$$

Example 3

$$f(x) = 4x^4 - 17x^2 + 4$$

Divide $f(x)$ by $(2x + 1)$, giving your answer in the form $f(x) = (2x + 1)(ax^3 + bx^2 + cx + d)$.

$$\begin{array}{r}
 2x+1 \overline{) \begin{array}{l} 2x^3 - x^2 - 8x + 4 \\ 4x^4 + 0x^3 - 17x^2 + 0x + 4 \\ \underline{4x^4 + 2x^3} \\ -2x^3 - 17x^2 \\ \underline{-2x^3 - x^2} \\ -16x^2 + 0x \\ \underline{-16x^2 - 8x} \\ 8x + 4 \\ \underline{8x + 4} \\ 0 \end{array}
 \end{array}$$

$$(2x+1)(2x^3 - x^2 - 8x + 4)$$

Example 4

Find the remainder when $2x^3 - 5x^2 - 16x + 10$ is divided by $(x - 4)$.

$$\begin{array}{r}
 x-4 \overline{) \begin{array}{l} 2x^3 - 5x^2 - 16x + 10 \\ 2x^3 - 8x^2 \\ \underline{2x^3 - 8x^2} \\ 3x^2 - 16x \\ 3x^2 - 12x \\ \underline{3x^2 - 12x} \\ -4x + 10 \\ \underline{-4x + 16} \\ -6 \end{array}
 \end{array}$$

c $-3x^3 + 50x - 8$ by $(x - 4)$

$$\begin{array}{r}
 x-4 \overline{) \begin{array}{l} -3x^3 - 12x + 2 \\ -3x^3 + 0x^2 + 50x - 8 \\ \underline{-3x^3 + 12x^2} \\ -12x^2 + 50x \\ \underline{-12x^2 + 48x} \\ 2x - 8 \\ \underline{2x - 8} \\ 0 \end{array}
 \end{array}
 \quad (x-4)(-3x^2 - 12x + 2)$$

14 Divide $x^4 - 16$ by $(x + 2)$.

$$\begin{array}{r}
 x^3 - 2x^2 + 4x - 8 \\
 x+2 \overline{) x^4 + 0x^3 + 0x^2 + 0x - 16} \\
 \underline{x^4 + 2x^3} \\
 -2x^3 + 0x^2 \\
 \underline{-2x^3 - 4x^2} \\
 4x^2 + 0x \\
 \underline{4x^2 + 8x} \\
 -8x - 16 \\
 \underline{-8x - 16} \\
 0
 \end{array}$$

Example 5

Show that $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$ by:

a algebraic division

b the factor theorem

b) If $(x - 2)$ is a factor then $x = 2$ will be a solution

$$f(2) = 0 \quad \therefore \quad 2^3 + 2^2 - 4 \times 2 - 4 = 8 + 4 - 8 - 4 = 0$$

Example 6

a Fully factorise $2x^3 + x^2 - 18x - 9$

b Hence sketch the graph of $y = 2x^3 + x^2 - 18x - 9$

a) $x = 3$

$$= 2 \times 3^3 + 3^2 - 18 \times 3 - 9$$

$$= \cancel{54} + \cancel{9} - \cancel{54} - \cancel{9}$$

$\approx$

$x = 3$ is a solution

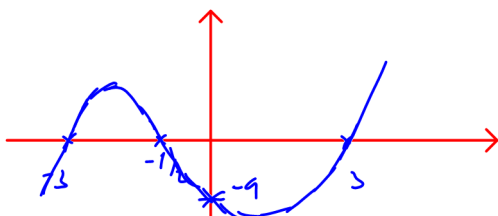
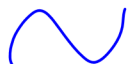
$\therefore (x - 3)$ is a factor

$$\begin{array}{r}
 2x^2 + 7x + 3 \\
 x-3 \overline{) 2x^3 + x^2 - 18x - 9} \\
 \underline{2x^3 - 6x^2} \\
 7x^2 - 18x \\
 \underline{7x^2 - 21x} \\
 3x - 9 \\
 \underline{3x - 9} \\
 0
 \end{array}$$

$$(x - 3)(2x^2 + 7x + 3)$$

$$(x - 3)(2x + 1)(x + 3)$$

b) $x = 3, \quad x = -1/2, \quad x = -3$



Example 7

Given that $(x + 1)$ is a factor of $4x^4 - 3x^2 + a$, find the value of a .

If $(x+1)$ is a factor then $x = -1$ is a solution

$$f(-1) = 0 \therefore 4(-1)^4 - 3(-1)^2 + a = 0$$

$$4 - 3 + a = 0 \quad a + 1 = 0$$

$$\boxed{a = -1}$$

11 Given that $(x + 1)$ and $(x - 2)$ are factors of $cx^3 + dx^2 - 9x - 10$, find the values of c and d .

$$f(-1) = 0 \therefore c(-1)^3 + d(-1)^2 - 9(-1) - 10 = 0$$

$$-c + d + 9 - 10 = 0$$

$$-c + d = 1 \quad (1)$$

$$f(2) = 0 \therefore c \times 2^3 + d \times 2^2 - 9 \times 2 - 10 = 0$$

$$8c + 4d - 18 - 10 = 0$$

$$8c + 4d = 28 \div 4 \quad 2c + d = 7 \quad (2)$$

$$-c + d = 1$$

$$-c + d = 1$$

$$-(2c + d = 7)$$

$$-3c + 0 = -6$$

$$\boxed{c = 2}$$

$$d = 1 + c$$

$$\boxed{d = 3}$$

Example 9

Prove that if $(x - p)$ is a factor of $f(x)$ then $f(p) = 0$.

$$\frac{f(x)}{x-p} = g(x)$$

$$f(x) = (x-p)g(x)$$

$$x = p \therefore f(p) = \cancel{(p-p)}^0 g(p) = 0$$

Example 11

The equation $kx^2 + 3kx + 2 = 0$, where k is a constant, has no real roots.

Prove that k satisfies the inequality $0 \leq k < \frac{8}{9}$

$$b^2 - 4ac < 0$$

$$(3k)^2 - 4k \times 2 < 0$$

$$9k^2 - 8k < 0$$

$$9k^2 - 8k = 0 \quad k(9k - 8) = 0 \quad k \neq 0 \quad \text{or} \quad k = \frac{8}{9}$$