

Example 3

Find the following integrals.

a $\int \cos(2x+3) dx$

b $\int e^{4x+1} dx$

c $\int \sec^2 3x dx$

a) Assume the integral is $I = \sin(2x+3)$

$$y = \sin(2x+3) \quad \frac{dy}{dx} = 2\cos(2x+3) \quad \text{2 times the integral but we have}$$

$$\int \cos(2x+3) dx = \frac{1}{2} \sin(2x+3) + c$$

$$b) \int e^{4x+1} dx = \frac{e^{4x+1}}{(4x+1)'} + c = \frac{1}{4} e^{4x+1} + c$$

$$c) \int \sec^2 3x dx = \frac{\tan 3x}{3} + c$$

Example 4

Find the following integrals:

a $\int \frac{1}{3x+2} dx$

b $\int (2x+3)^4 dx$

a) $= \frac{1}{3} \ln|3x+2| + c$

b) $\frac{1}{5} (2x+3)^5 \times \frac{1}{2} + c$

$$= \frac{1}{10} (2x+3)^5 + c$$

7 Given $\int_{e^2}^{e^8} \frac{1}{kx} dx = \frac{1}{4}$, find the value of k .

$$\left[\frac{1}{k} \ln kx \right]_{e^2}^{e^8} = \frac{1}{4} \Rightarrow \left(\frac{1}{k} \ln k e^8 \right) - \left(\frac{1}{k} \ln k e^2 \right) = \frac{1}{4}$$

$$\frac{1}{k} (\cancel{\ln k} + \ln e^8 - \cancel{\ln k} - \ln e^2) = \frac{1}{4}$$

$$\frac{1}{k} (8-2) = \frac{1}{4} \Rightarrow \frac{6}{k} = \frac{1}{4} \Rightarrow \boxed{k=24}$$

8 Given $\int_{\frac{\pi}{4k}}^{\frac{\pi}{3k}} (1 - \pi \sin kx) dx = \pi(7 - 6\sqrt{2})$,

find the exact value of k .

(7 marks)

$$\left[x + \frac{\pi}{k} \cos kx \right]_{\frac{\pi}{4k}}^{\frac{\pi}{3k}} = \pi(7 - 6\sqrt{2})$$

$$\left(\frac{\pi}{3k} + \frac{\pi}{k} \cos\left(k \cdot \frac{\pi}{3k}\right) \right) - \left(\frac{\pi}{4k} + \frac{\pi}{k} \cos\left(k \cdot \frac{\pi}{4k}\right) \right) = \pi(7 - 6\sqrt{2})$$

$$\left(\frac{1}{3k} + \frac{1}{k} \cos\left(\frac{\pi}{3}\right) \right) - \left(\frac{1}{4k} + \frac{1}{k} \cos\left(\frac{\pi}{4}\right) \right) = 7 - 6\sqrt{2}$$

$$\frac{5}{6k} - \frac{2 + \sqrt{2}}{4k} = 7 - 6\sqrt{2} \quad (\times 12k)$$

$$10 - 6 - 3\sqrt{2} = (7 - 6\sqrt{2}) 12k \quad \text{solve for } k$$

$$\frac{4 - 3\sqrt{2}}{7 - 6\sqrt{2}} = 12k \quad 12k = \frac{4 - 3\sqrt{2}}{7 - 6\sqrt{2}} \quad k = \frac{4 - 3\sqrt{2}}{276}$$

Example 5

Find $\int \tan^2 x dx$

we don't know that integral
we know $\int \sec^2 x dx = \tan x + c$

$$\sec^2 x = \tan^2 x + 1 \Rightarrow \tan^2 x = \sec^2 x - 1$$

$$\int (\sec^2 x - 1) dx = \tan x - x + c$$

Example 6

Show that $\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sin^2 x dx = \frac{\pi}{48} + \frac{1 - \sqrt{2}}{8}$

The rule: when you see integral of $\sin^2 x$ or $\cos^2 x$
you need to think of $\cos 2x$ (double angle)

$$\cos 2x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) dx = \left[\frac{1}{2} x - \frac{1}{4} \sin 2x \right]_{\frac{\pi}{12}}^{\frac{\pi}{8}}$$

$$\left(\frac{1}{2} \times \frac{\pi}{8} - \frac{1}{4} \sin\left(2 \cdot \frac{\pi}{8}\right) \right) - \left(\frac{1}{2} \times \frac{\pi}{12} - \frac{1}{4} \sin\left(2 \cdot \frac{\pi}{12}\right) \right)$$

$$\left(\frac{1}{16} \pi - \frac{\sqrt{2}}{8} \right) - \left(\frac{1}{24} \pi - \frac{1}{8} \right) = \dots$$

Find:

a $\int \sin 3x \cos 3x dx$

b $\int (\sec x + \tan x)^2 dx$

a) $\sin 2a = 2 \sin a \cos a$

$\therefore \sin 3x \cos 3x = \frac{1}{2} \sin 6x$

$\frac{1}{2} \int \sin 6x dx = -\frac{1}{12} \cos 6x + c$

b) $\int (\sec^2 x + \tan^2 x + 2 \sec x \tan x) dx$

$\int (\sec^2 x + \sec^2 x - 1 + 2 \sec x \tan x) dx$

$y = \sec x \Rightarrow \frac{dy}{dx} = \sec x \tan x$

$\int (2 \sec^2 x - 1 + 2 \sec x \tan x) dx$

$= 2 \tan x - x + 2 \sec x + c$

4 Find the exact value of each of the following:

a $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin^2 x \cos^2 x} dx$ b $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin x - \csc x)^2 dx$ c $\int_0^{\frac{\pi}{4}} \frac{(1 + \sin x)^2}{\cos^2 x} dx$ d $\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \frac{\sin 2x}{1 - \sin^2 2x} dx$

$\int \frac{1}{(\sin x \cos x)^2} dx = \int \frac{1}{\left(\frac{1}{2} \sin 2x\right)^2} dx$

$= \int 4 (\sin 2x)^{-2} dx$

$= 4x \frac{(-1)(\sin 2x)^{-1}}{2 \cos 2x} + c$

$= \frac{-4}{2 \sin 2x \cos 2x} + c$

$= \frac{-4}{\sin 4x} + c$

b $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin x - \csc x)^2 dx = \int (\sin^2 x + \csc^2 x - 2 \sin x \csc x) dx$

$= \int (\sin^2 x + \csc^2 x - 2 \frac{\sin x}{\sin x}) dx$

$\int (\sin^2 x) dx = \int \left(\frac{1}{2} - \frac{1}{2} \cos 2x\right) dx = \frac{1}{2} x - \frac{1}{4} \sin 2x$

$\int \csc^2 x dx = -\cot x$

$\frac{1}{2} x - \frac{1}{4} \sin 2x - \cot x - 2x$

Reverse chain rule

Example

8

Find

a $\int \frac{2x}{x^2+1} dx$

b $\int \frac{\cos x}{3+2\sin x} dx$

a) To be able to realize that chain rule can be used. The derivative of denominator should be similar to numerator.

we deal only with $\int \frac{1}{x^2+1} dx \Rightarrow$ Assume that $I = \ln|x^2+1|$

Differentiation - to see if you can get the original

$y = \ln|x^2+1| \quad \frac{dy}{dx} = \frac{1}{x^2+1} \times 2x$ that's the original

$\therefore \int \left(\frac{2x}{x^2+1}\right) dx = \ln(x^2+1) + C$

b $\int \frac{\cos x}{3+2\sin x} dx$

We deal with $\int \frac{1}{3+2\sin x} dx \rightarrow$ Assume $I = \ln|3+2\sin x|$

go back $y = \ln|3+2\sin x| \quad \frac{dy}{dx} = \frac{1}{3+2\sin x} \times 2\cos x$

it's twice the original

$\therefore \int \frac{\cos x}{3+2\sin x} dx = \frac{1}{2} \ln|3+2\sin x| + C$

3) $\int \frac{\sin x}{\cos x} dx = \int \frac{\sin x}{\cos x} dx$

we deal with $\int \frac{1}{\cos x} dx \rightarrow$ Assume $I = \ln|\cos x|$

go back $y = \ln|\cos x| \quad \frac{dy}{dx} = \frac{1}{\cos x} (-1 \sin x)$

$\therefore \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C$

Example 9

Find:

a $\int 3 \cos x \sin^2 x \, dx$

b $\int x(x^2 + 5)^3 \, dx$

a) we deal only with $\int \sin^2 x \, dx$ Assume $I = \sin^2 x$ go backwards $\frac{dy}{dx} = 2 \sin^2 x \cos x$ that's the original

$$\int 3 \cos x \sin^2 x \, dx = \sin^3 x + C$$

b $\int x(x^2 + 5)^3 \, dx$

deal only with $\int (x^2 + 5)^3 \, dx$ Assume $I = (x^2 + 5)^4$ go backwards $\frac{dy}{dx} = 4(x^2 + 5)^3 \times 2x = 8x(x^2 + 5)^3$
11s 8 times greater than original.

$$\therefore \int x(x^2 + 5)^3 \, dx = \frac{1}{8} (x^2 + 5)^4 + C$$

Example 10Use integration to find $\int \frac{\operatorname{cosec}^2 x}{(2 + \cot x)^3} \, dx$ we deal with $\int (2 + \cot x)^3 \, dx$ Assume $I = (2 + \cot x)^2$ go backwards: $\frac{dy}{dx} = -2(2 + \cot x)^2 \times (-\operatorname{cosec}^2 x)$

$$= \frac{2 \operatorname{cosec}^2 x}{(2 + \cot x)^2} \quad 2 \times \text{the original}$$

$$\int \frac{\operatorname{cosec}^2 x}{(2 + \cot x)^3} \, dx = \frac{1}{2} (2 + \cot x)^2 + C$$

Example 11Given that $\int_0^\theta 5 \tan x \sec^4 x \, dx = \frac{15}{4}$ where $0 < \theta < \frac{\pi}{2}$, find the exact value of θ .we deal with $\int \sec^4 x \, dx$ Assume $I = \sec^5 x$ $\frac{dy}{dx} = 5 \sec^4 x (\sec x \tan x) = 5 \sec^5 x \tan x$ It doesn't give me the original. Assumption was wrong

$$\therefore \int \sec^4 x dx \quad \text{Assume } I = \sec^4 x$$

$$\frac{dy}{dx} = 4 \sec^3 x (\sec x \tan x)$$

$$= 4 \tan x \sec^4 x \quad \text{its } \frac{5}{4} \text{ the original}$$

$$\int 5 \tan x \sec^4 x dx = \frac{5}{4} \sec^4 x + C$$

$$\text{ii } \int \frac{\sin x \cos x}{\sqrt{\cos 2x + 3}} dx \Rightarrow$$

$$\int (\cos 2x + 3)^{-1/2} dx \quad \text{Assume } I = (\cos 2x + 3)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2} (\cos 2x + 3)^{-1/2} \times (-2 \sin 2x)$$

$$= -\frac{\sin 2x}{\sqrt{\cos 2x + 3}} = -\frac{2 \sin x \cos x}{\sqrt{\cos 2x + 3}} \quad (-2) \times \text{times greater than original}$$

$$\int \frac{\sin x \cos x}{\sqrt{\cos 2x + 3}} dx = -\frac{1}{2} (\cos 2x + 3)^{1/2} + C$$

$$\text{d } \int_0^{\frac{\pi}{4}} \sec^2 x e^{4 \tan x} dx$$

$$\int e^{4 \tan x} dx \quad \text{Assume } I = e^{4 \tan x}$$

$$\frac{dy}{dx} = e^{4 \tan x} \times (4 \tan x)'$$

$$= 4 \sec^2 x e^{4 \tan x} \quad 4 \times \text{original} \therefore$$

$$\therefore \int \sec^2 x e^{4 \tan x} dx = \frac{1}{4} e^{4 \tan x} + C$$

$$\text{5 Given that } \int_0^{\theta} 4 \sin 2x \cos^4 2x dx = \frac{4}{5} \text{ where } 0 < \theta < \pi, \text{ find the exact value of } \theta.$$

$$\int \cos^4 2x dx \quad \text{Assume } I = \cos^5 2x$$

$$\frac{dy}{dx} = 5 \cos^4 2x \times (-2 \sin 2x)$$

$$= -10 \sin 2x \cos^4 2x$$

$$\therefore \int 4 \sin 2x \cos^4 2x dx = -\frac{4}{10} \cos^5 2x + C$$