

CHAPTER 7

7.1 Algebraic fractions

- $3x^3 + 4x^2 - 2x$
 - $4 - 8x + 3x^2$
 - $-4x^4 + 2x^3 - x^2$
- $x + 3$
 - $x - 1$
 - $2x - 1$
- $\frac{2x+1}{x+3}$
 - $\frac{3x-2}{2x+3}$
 - $\frac{3x-2}{x-4}$
- $\frac{(3x-5)(5x+3)}{x(3x-5)} = \frac{5x+3}{x} = 5 + \frac{3}{x}$
- $k = -15$
- $k = 2$
- $a = 2, b = 3, c = 2, p = 2, q = -5$

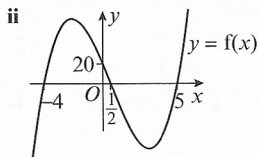
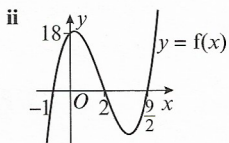
7.2 Dividing polynomials

- $(x-1)(2x^2-3x+5)$
 - $(x+2)(3x^2+2x-1)$
 - $(x-3)(2x^2+7x+4)$
 - $(x+4)(4x^2-3x+1)$
- $3x^3+2x^2-5x-3$
 - $2x^3-5x^2+3x-2$
 - $3x^3+x^2-5x+1$
 - $2x^3-3x^2-7x-3$
- $2x^2+4x-4$
 - $3x^2-2x+6$
 - $4x^2+8x+5$
 - $2x^2-3x+15$
- -44
 - 25
 - -80
- 3
- -210
 - $(x-2)(2x+3)(3x-5)$
- $k = 15$
 - $(x+3)(2x^2-3x+5)$
 - One solution $x = -3$ when $x+3 = 0$.
For $2x^2-3x+5 = 0$, $b^2-4ac = -31 < 0$,
so no other real solutions.

- 43
 - $(x-2)(3x^2+16x+24)+43$
- $(x-1)(10x^2-19x-15)$
 - $(x-1)(2x-5)(5x+3)$
 - $x = 1, x = \frac{5}{2}, x = -\frac{3}{5}$

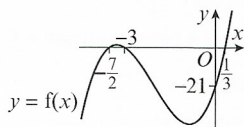
7.3 The factor theorem

- $f(-1) = 0$
 - $f(-2) = 0$
 - $f(3) = 0$
 - $f(4) = 0$
- $f(2) = 0, (x-2)(x+3)(2x-1)$
 - $f(-3) = 0, (x+3)(x+5)(2x+1)$
 - $f(1) = 0, (x-1)(3x-2)(2x+3)$
 - $f(-4) = 0, (x+4)(5x-3)(3x+2)$
- $(x-1)(x-3)(x+6)$
 - $(x+2)(x+5)(2x-1)$
 - $(x+3)(x-4)(3x+5)$
- $(x-2)(x+1)(2x-9)$
 - $(x-5)(x+4)(2x-1)$



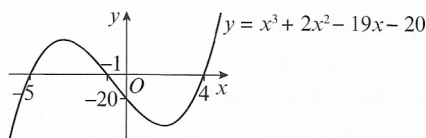
c i $(x+3)(2x+7)(3x-1)$

ii



- $x = 3, x = -\frac{3}{2}, x = \frac{4}{3}$
- $f(-4) = 0$
 - $(x+4)(3x-2)^2$
- $(x-1)(x-2)(2x+3)$
- $g(-1) = -1 + 2 + 19 + k = 0 \Rightarrow k = -20$
 - $(x+1)(x-4)(x+5)$

c



- $p(-3) = 0$
 - $(x+3)(5x-2)^2$
- $x = -3$ and $x = \frac{2}{5}$

7.4 Mathematical proof

- LHS $= (x-4)(2x^2+15x+18) = 2x^3+7x^2-42x-72$ = RHS
- Using Pythagoras' theorem:
 $PQ^2 = 16^2 + 12^2 = 400$, $QR^2 = 18^2 + 6^2 = 360$
 $PR^2 = 2^2 + 6^2 = 40$
 $QR^2 + PR^2 = 360 + 40 = 400 = PQ^2$
As $PQ^2 = QR^2 + PR^2$ using Pythagoras, triangle PQR is right-angled.
- At point of intersection, $kx = 4x^2 - 5x + 4$, so
 $4x^2 - x(5+k) + 4 = 0$
Only one real root, so $b^2 - 4ac = 0$ gives $k^2 + 10k - 39 = 0$
 $\Rightarrow (k+13)(k-3) = 0 \Rightarrow k = 3$ and $k = -13$, so the lines with equations $y = 3x$ and $y = -13x$ are tangents to the curve.
- If A, B and C lie on the same straight line, the gradients of the line segments AB, BC and AC are equal.

$$\text{Gradient of } AB = \frac{1-3}{2+2} = -\frac{2}{4} = -\frac{1}{2}$$

$$\text{Gradient of } BC = \frac{-5-1}{14-2} = -\frac{6}{12} = -\frac{1}{2}$$

As gradient of AB = gradient of BC and B is a common point, then A, B and C are collinear.

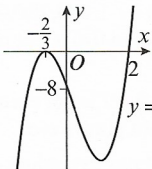
- Completing the square,
 $x^2 - 6x + 10 = (x-3)^2 - 9 + 10 = (x-3)^2 + 1$
As $(x-3)^2 \geq 0$ for all values of x , then $(x-3)^2 + 1 \geq 1$
and $x^2 - 6x + 10 > 0$ for all values of x .
- LHS $= x^2 + 2xy + y^2 - (x^2 - 2xy + y^2) = 4xy$ = RHS
- If first odd number is $2n+1$, next consecutive odd number is $2n+3$.
 $(2n+3)^2 - (2n+1)^2 = 4n^2 + 12n + 9 - (4n^2 + 4n + 1)$
 $= 8n + 8 = 8(n+1)$
So the difference between the squares of two consecutive odd numbers is a multiple of 8.
- Let odd number be $2n+1$.
 $(2n+1)^3 - (2n+1)^2 = 8n^3 + 12n^2 + 6n + 1 - (4n^2 + 4n + 1)$
 $= 8n^3 + 8n^2 + 2n$
As $8n^3 + 8n^2 + 2n = 2(4n^3 + 4n^2 + n)$, the difference between the cube and the square of an odd number is even.
- No real roots means $b^2 - 4ac < 0$.
Solving $(k-3)^2 - 4 \times 1 \times (3-2k) = 0$ gives
 $k^2 + 2k - 3 = 0 \Rightarrow (k+3)(k-1) = 0$
Critical values are -3 and 1 , so for no real roots $-3 < k < 1$.

7.5 Methods of proof

- 1 a $n = 5: 5^2 + 5 + 17 = 47; n = 6: 6^2 + 6 + 17 = 59$
 $n = 7: 7^2 + 7 + 17 = 73; n = 8: 8^2 + 8 + 17 = 89$
 $n = 9: 9^2 + 9 + 17 = 107; n = 10: 10^2 + 10 + 17 = 127$
 Proof by exhaustion, so for $5 \leq n \leq 10$, $f(n)$ is prime.
- b When $n = 17: 17^2 + 17 + 17 = 17(17 + 1 + 1)$, divisible by 17 so not prime. Counter-example, so $f(n)$ is not prime for all positive integers, n .
- 2 2 and 3 are both prime numbers. $3 - 2 = 1$ is not even, so the difference of two prime numbers is not always even.
- 3 a $(1 + 2x)^2 - 4x^2 - 1 = 4x > 0$ for all positive values of x .
 So $(1 + 2x)^2 - 4x^2 - 1 > 0 \Rightarrow (1 + 2x)^2 > 1 + 4x^2$
- b If $x = -1$, $(1 + 2x)^2 = 1$ and $1 + 4x^2 = 5$, counter-example, so $(1 + 2x)^2 > 1 + 4x^2$ is not true for all values of x .
- 4 a Cancelling a factor of x is not valid when $x = 0$.
- b $x = 0$
- 5 When $x = -2$, $(-2 + 2)^2 = 0$ and $(-2)^2 = 4$, so $(x + 2)^2 > x^2$ is not true when $x = -2$.
- 6 $n = 1: 9 > 4; n = 2: 25 > 8; n = 3: 49 > 16; n = 4: 81 > 32$
 $n = 5: 121 > 64; n = 6: 169 > 128$
- 7 Sometimes true. For example, when $x = 4$, $4^2 = 16$ which is greater than 4, so true. But when $x = \frac{1}{2}$, $(\frac{1}{2})^2 = \frac{1}{4}$, which is not greater than $\frac{1}{2}$, so false.
- 8 $(x + 5)^2 - 4x - 9 = x^2 + 6x + 16 = (x + 3)^2 + 7 \geq 0$ for all real values of x , proved by completing the square.
 So $(x + 5)^2 - 4x - 9 \geq 0 \Rightarrow (x + 5)^2 \geq 4x + 9$
- 9 Proof by exhaustion:
 $n = 2: 2^2 + 2 = 6$ not divisible by 4
 $n = 3: 3^2 + 2 = 11$ not divisible by 4
 $n = 4: 4^2 + 2 = 18$ not divisible by 4
 $n = 5: 5^2 + 2 = 27$ not divisible by 4
 $n = 6: 6^2 + 2 = 38$ not divisible by 4
 $n = 7: 7^2 + 2 = 51$ not divisible by 4

Problem solving: Set A

- B a -24 b $(x - 4)(x^2 - 5x - 18) - 24$
- c $(x + 2)(x - 3)(x - 8)$
- d Proof by exhaustion:
 $n = 4: p(4) = 4^3 - 9 \times 4^2 + 2 \times 4 + 48 = -24 < 0$
 $n = 5: p(5) = 5^3 - 9 \times 5^2 + 2 \times 5 + 48 = -42 < 0$
 $n = 6: p(6) = 6^3 - 9 \times 6^2 + 2 \times 6 + 48 = -48 < 0$
 $n = 7: p(7) = 7^3 - 9 \times 7^2 + 2 \times 7 + 48 = -36 < 0$
 So for $3 < n < 8$, $x^3 - 9x^2 + 2x + 48 < 0$
- S a $(2x + 1)(x^2 - 8x + 20)$
- b One root when $2x + 1 = 0$, i.e. $x = -\frac{1}{2}$
 Using discriminant for $x^2 - 8x + 20$,
 $b^2 - 4ac = 64 - 4 \times 1 \times 20 = -16 < 0$ indicates no real roots so $p(x) = 0$ has exactly one real root.
- c $2x^3 - 15x^2 + 32x + 20 = (2x + 1)(x^2 - 8x + 20)$
 $2x + 1 > 0$ when $x > -\frac{1}{2}$, so $2x + 1 > 0$ for all real positive values of x .
 Completing the square for $x^2 - 8x + 20$ gives
 $(x - 4)^2 + 4$. As $(x - 4)^2 \geq 0$ for all real values of x ,
 $(x - 4)^2 + 4 \geq 4$ for all real values of x and
 $(2x + 1)(x^2 - 8x + 20) > 0$ for all real positive values of x .

- G a $a = 9, b = 12, c = 4, d = -2$
- b $p(x) = (x - 2)(9x^2 + 12x + 4)$. One real root when $x = 2$.
 Using discriminant for $9x^2 + 12x + 4$,
 $b^2 - 4ac = 144 - 4 \times 9 \times 4 = 0$ indicates one real root, so
 $p(x) = 0$ has exactly two real roots.
- c 
- d $\{x : x < -\frac{2}{3}\} \cup \{x : -\frac{2}{3} < x < 2\}$

Problem solving: Set B

- B a $f(-1) = -1 - 1 - p + q = 0$, which gives $q - p = 2$
- b $f(-3) = -27 - 9 - 3p + q = 0$, which gives $q - 3p = 36$
- c $p = -17$ and $q = -15$
- d $f(x) = (x + 1)(x + 3)(x - 5)$
- S a $f(-2) = -16 - 4 - 2p + q = 0$, gives $q - 2p - 20 = 0$
- b $p = -13$ and $q = -6$
- c $f(x) = (x + 2)(x - 3)(2x + 1)$
- G a $f(2) = 8 + (p + 4) \times 4 + 16 + q = 0$, which gives
 $4p + q + 40 = 0$
- b $f(-p) = -p^3 + p^3 + 4p^2 - 8p + q = 0$.
 Simplify to obtain $4p^2 - 8p + q = 0$
- c $p = 5$ and $q = -60$
- d $f(x) = (x - 2)(x + 5)(x + 6)$