

01.10

4 Find $\frac{dy}{dx}$ given that:

a $y = \sin 2x + \cos 3x$

b $y = 2 \cos 4x - 4 \cos x + 2 \cos 7x$

c $y = x^2 + 4 \cos 3x$

d $y = \frac{1 + 2x \sin 5x}{x}$

a) $\frac{dy}{dx} = 2 \cos 2x - 3 \sin 3x$

b) $\frac{dy}{dx} = -8 \sin 4x + 4 \sin x - 14 \sin 7x$

d) $y = \frac{1}{x} + 2 \sin 5x$

$$\frac{dy}{dx} = -\frac{1}{x^2} + 10 \cos 5x$$

10 Prove, from first principles, that the derivative of $\sin x$ is $\cos x$.

You may assume the formula for $\sin(A+B)$ and that as $h \rightarrow 0$, $\frac{\sin h}{h} \rightarrow 1$ and $\frac{\cos h - 1}{h} \rightarrow 0$.

(5 marks)

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = \sin x$$

$$f(x+h) = \sin(x+h)$$

$$= \sin x \cos h + \cos x \sin h$$

$$\lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin x (\cos h - 1)}{h} + \lim_{h \rightarrow 0} \frac{\cos x \sin h}{h}$$

Small angle approx..

$$\sin h \approx h \quad \cos h \approx 1 - \frac{1}{2} h^2$$

$$= \lim_{h \rightarrow 0} \frac{\sin x \left(1 - \frac{1}{2} h^2 - 1\right)}{h} + \lim_{h \rightarrow 0} \frac{\cos x h}{h}$$
$$= \cos x$$

$$y = e^{sx}$$

$$\frac{dy}{dx} = s e^{sx}$$

$$y = \ln x$$

$$\frac{dy}{dx} = \frac{1}{x}$$

Example

4

Show that the derivative of a^x is $a^x \ln a$.

$$y = a^x \quad \ln y = x \ln a$$

$$y = e^{x \ln a} \quad \frac{dy}{dx} = \ln a \cdot e^{x \ln a}$$

$$= \ln a \cdot a^x$$

$$\frac{dy}{dx} = a^x \ln a$$

$$y = 2^x \quad \frac{dy}{dx} = 2^x \ln 2$$

- 5 A curve has the equation $y = e^{2x} - \ln x$. Show that the equation of the tangent at the point with x -coordinate 1 is

$$y = (2e^2 - 1)x - e^2 + 1$$

(6 marks)

$$\frac{dy}{dx} = 2e^{2x} - \frac{1}{x} \quad x=1 \quad \frac{dy}{dx} = 2e^2 - 1$$

$$x=1, \quad y = e^2 - \frac{0}{1} \quad (1, e^2)$$

$$y - e^2 = (2e^2 - 1)(x - 1)$$

$$y - e^2 = (2e^2 - 1)x - 2e^2 + 1 \Rightarrow y = (2e^2 - 1)x - e^2 + 1$$

- 10 $f(x) = e^{2x} - \ln(x^2) + 4, x > 0$

a Find $f'(x)$.

(3 marks)

The curve with equation $y = f(x)$ has a gradient of 2 at point P . The x -coordinate of P is a .

b Show that $a(e^{2a} - 1) = 1$.

(2 marks)

$$a) f(x) = e^{2x} - 2 \ln x + 4$$

$$f'(x) = 2e^{2x} - \frac{2}{x}$$

$$b) 2e^{2a} - \frac{2}{a} = 2 \quad 2ae^{2a} - 2 = 2a \quad \div 2$$

$$ae^{2a} - 1 = a \Rightarrow a(e^{2a} - 1) = 1 \quad \checkmark$$

2 Differentiate:

a $e^{\cos x}$

b $\cos(2x-1)$

c $\sqrt{\ln x}$

d $(\sin x + \cos x)^5$

a) $u = \cos x$ $y = e^u$ $\frac{dy}{du} = e^u = e^{\cos x}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
 $= -\sin x e^{\cos x}$

b) $\frac{dy}{dx} = -\sin(2x-1) (2x-1)'$
 $= -2\sin(2x-1)$

c) $y = \sqrt{\ln x}$ $\frac{dy}{dx} = \frac{1}{2} (\ln x)^{-1/2} (\ln x)'$
 $= \frac{1}{2} (\ln x)^{-1/2} \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$

d $(\sin x + \cos x)^5$

$\frac{dy}{dx} = 5(\sin x + \cos x)^4 (\sin x + \cos x)'$
 $= 5(\sin x + \cos x)^4 (\cos x - \sin x)$

e $\sin(3x^2 - 2x + 1)$

f $\ln(\sin x)$

g $2e^{\cos 4x}$

h $\cos(e^{2x} + 3)$

e) $\frac{dy}{dx} = \cos(3x^2 - 2x + 1) (6x - 2)$
 $= (6x - 2) \cos(3x^2 - 2x + 1)$

f) $\frac{dy}{dx} = \frac{1}{\sin x} \times \cos x = \frac{\cos x}{\sin x}$

g) $\frac{dy}{dx} = 2(\cos 4x)' e^{\cos 4x}$
 $= -8\sin 4x e^{\cos 4x}$

h) $\frac{dy}{dx} = -\sin(e^{2x} + 3) \times 2e^{2x}$
 $= -2e^{2x} \sin(e^{2x} + 3)$

11 Differentiate:

a $\sin^2 3x$

b $e^{(x+1)^2}$

c $\ln(\cos x)^2$

a) $\frac{dy}{dx} = 2\sin 3x \times (\sin 3x)'$
 $= 2\sin 3x \times 3\cos 3x$
 $= 6\sin 3x \cos 3x$

b) $y = e^{(x^2+1)x+1}$
 $\frac{dy}{dx} = (2x+1)e^{(x^2+1)x+1}$

c) $y = \ln(\cos^2 x)$

$$c) y = \ln(\cos^2 x)$$

$$\frac{dy}{dx} = \frac{1}{\cos^2 x} \times (\cos^2 x)' \rightarrow u^2 \rightarrow 2u \left(\frac{1}{dx} \right)$$

$$= \frac{1}{\cos^2 x} \cdot 2 \cos x (-\sin x)$$

$$= \frac{-2 \cos x \sin x}{\cos^2 x} = -\frac{2 \sin x}{\cos x}$$

$$y = f(x) g(x) \quad \frac{dy}{dx} = f'(x) g(x) + f(x) g'(x)$$

Example 11

Given that $y = e^{4x} \sin^2 3x$, show that $\frac{dy}{dx} = e^{4x} \sin 3x (A \cos 3x + B \sin 3x)$, where A and B are constants to be determined.

$$\frac{dy}{dx} = u e^{4x} \sin^2 3x + e^{4x} (2 \sin 3x \times 3 \cos 3x)$$

$$= u e^{4x} \sin^2 3x + 6 e^{4x} (\sin 3x \cos 3x)$$

$$= e^{4x} \sin 3x (u \sin 3x + 6 \cos 3x)$$

6 A curve C has equation $y = x^2 \cos(x^2)$. Find the equation of the tangent to the curve C at the point $P\left(\frac{\sqrt{\pi}}{2}, \frac{\pi\sqrt{2}}{8}\right)$ in the form $ax + by + c = 0$ where a , b and c are exact constants. (7 marks)

$$\frac{dy}{dx} = 2x \cos(x^2) + x^2 (-\sin(x^2) \times 2x)$$

$$= 2x \cos(x^2) - 2x^3 \sin(x^2)$$

$$x = \frac{\sqrt{\pi}}{2} \Rightarrow \frac{dy}{dx} = 2 \frac{\sqrt{\pi}}{2} \cos\left(\left(\frac{\sqrt{\pi}}{2}\right)^2\right) - 2 \left(\frac{\sqrt{\pi}}{2}\right)^3 \sin\left(\left(\frac{\sqrt{\pi}}{2}\right)^2\right)$$

$$= \sqrt{\pi} \frac{\sqrt{2}}{2} - \frac{2}{8} \pi \sqrt{\pi} \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2\pi}}{2} - \frac{\pi\sqrt{2\pi}}{8} = \frac{4\sqrt{2\pi} - \pi\sqrt{2\pi}}{8}$$

$$y - \frac{\pi\sqrt{2}}{8} = \frac{4\sqrt{2\pi} - \pi\sqrt{2\pi}}{8} \left(x - \frac{\sqrt{\pi}}{2} \right)$$

$$8y - \pi\sqrt{2} = (4\sqrt{2\pi} - \pi\sqrt{2\pi}) \left(x - \frac{\sqrt{\pi}}{2} \right)$$

$$8y - \pi\sqrt{2} = (4\sqrt{2\pi} - \pi\sqrt{2\pi})x - 4\sqrt{2\pi} \frac{\sqrt{\pi}}{2} + \pi\sqrt{2\pi} \frac{\sqrt{\pi}}{2}$$

$$8y - \pi\sqrt{2} = (4\sqrt{2\pi} - \pi\sqrt{2\pi})x - 2\pi\sqrt{2} + \frac{\pi^2\sqrt{2}}{2}$$

$$8y - \pi r_2 = (4\sqrt{2\pi} - \pi\sqrt{2\pi})x - 2\pi r_2 + \frac{\pi^2}{2} r_2$$

$$(4\sqrt{2\pi} - \pi\sqrt{2\pi})x - 8y - 2\pi r_2 + \pi\sqrt{2} + \frac{\pi^2}{2} r_2 = 0$$

$$(4\sqrt{2\pi} - \pi\sqrt{2\pi})x - 8y - \pi r_2 + \frac{\pi^2}{2} r_2 = 0$$

9 Differentiate with respect to x :

a $(2\sin x - 3\cos x) \ln 3x$ a b

b $x^4 e^{7x-3}$

a) $\frac{dy}{dx} = (2\cos x + 3\sin x) \ln 3x + (2\sin x - 3\cos x) \frac{1}{x}$ ~~$\times 3$~~

b) $\frac{dy}{dx} = 4x^3 e^{7x-3} + x^4 (7e^{7x-3})$
 $= x^3 e^{7x-3} (4 + 7x)$

a $y = e^x \sin^2 x \cos x$

$$y = (e^x \sin^2 x) \cos x$$

$$= (e^x \sin^2 x + e^x 2 \sin x \cos x) \cos x + (e^x \sin^2 x)(-\sin x)$$

$$= e^x \sin^2 x \cos x + 2e^x \sin x \cos^2 x - e^x \sin^3 x$$

$$= e^x \sin x (\sin x \cos x + 2 \cos^2 x - \sin^2 x)$$

Quotient rule

$$y = \frac{f(x)}{g(x)} \quad \frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Example 12

Given that $y = \frac{x}{2x+5}$ find $\frac{dy}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1(2x+5) - x(2)}{(2x+5)^2} \\ &= \frac{2x+5-2x}{(2x+5)^2} = \frac{5}{(2x+5)^2} \end{aligned}$$

A curve C with equation $y = \frac{\sin x}{e^{2x}}$, $0 < x < \pi$, has a stationary point at P . Find the coordinates of P . Give your answer to 3 significant figures.

$$\frac{dy}{dx} = \frac{\cos x \times e^{2x} - \sin x (2e^{2x})}{(e^{2x})^2}$$

$$= \frac{\cos x - 2 \sin x}{e^{2x}} = 0 \quad \cos x - 2 \sin x = 0$$

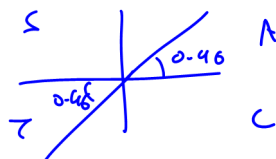
$$\frac{\cos x}{\sin x} = \frac{2 \sin x}{\sin x} \Rightarrow 1 = 2 \tan x$$

$$\tan x = \frac{1}{2}$$

$$x = 0.464, \pi + 0.46$$

$$y = \frac{\sin(0.464)}{e^{(1 \times 0.46)}} = 0.177$$

$$(0.464, 0.177) \text{ (3 s.f.)}$$



2 Differentiate:

a $\frac{e^{4x}}{\cos x}$

b $\frac{\ln x}{x+1}$

c $\frac{e^{-2x} + e^{2x}}{\ln x}$

d $\frac{(e^x + 3)^3}{\cos x}$

e $\frac{\sin^2 x}{\ln x}$

a) $\frac{dy}{dx} = \frac{4e^{4x} \cos x + e^{4x} \sin x}{\cos^2 x}$

b) $\frac{dy}{dx} = \frac{\frac{1}{x}(x+1) - \ln x (1)}{(x+1)^2}$

$$= \frac{1 + \frac{1}{x} - \ln x}{(x+1)^2}$$

c) $\frac{dy}{dx} = \frac{(-2e^{-2x} + 2e^{2x}) \ln x - (e^{-2x} + e^{2x}) \frac{1}{x}}{\ln^2 x}$

d) $\frac{dy}{dx} = \frac{3(e^x + 3)^2 \times e^x \cos x - (e^x + 3)^3 \times (-\sin x)}{\cos^2 x}$

$$= \frac{3e^x (e^x + 3)^2 \cos x + (e^x + 3)^3 \sin x}{\cos^2 x}$$

e) $\frac{dy}{dx} = \frac{(1 \sin x \cos x) \ln x - \frac{1}{x} \sin^2 x}{(\ln x)^2}$

8 The curve C has equation $x = \frac{e^y}{3+2y}$

a Find the coordinates of the point P where the curve cuts the x -axis. (1 mark)

b Find an equation of the normal to the curve at P , giving your answer in the form $y = mx + c$, where m and c are integers to be found. (6 marks)

$$a) \quad y=0 \quad \therefore \quad x = \frac{e^0}{3+0} \Rightarrow x = \frac{1}{3}$$

$$\left(\frac{1}{3}, 0\right)$$

$$b) \quad \frac{dx}{dy} = \frac{e^y(3+2y) - 2e^y}{(3+2y)^2} = \frac{e^y + 2y}{(3+2y)^2}$$

$$\frac{dy}{dx} = \frac{(3+2y)^2}{e^y + 2y} \quad \left(\frac{1}{3}, 0\right)$$

$$y=0 \therefore \frac{dy}{dx} = \frac{(3+0)^2}{e^0 + 0} = \frac{9}{1} \quad \text{normal: } m = -\frac{1}{9}$$

$$y - 0 = -\frac{1}{9} \left(x - \frac{1}{3}\right) \Rightarrow y = -\frac{1}{9}x + \frac{1}{27}$$

Example 14

If $y = \tan x$, find $\frac{dy}{dx}$

$$y = \frac{\sin x}{\cos x} \quad \text{quotient rule}$$

$$\frac{dy}{dx} = \frac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$y = \tan x \quad \frac{dy}{dx} = \sec^2 x$$

$$y = \tan 5x \quad \frac{dy}{dx} = 5 \sec^2 5x$$

$$y = \cot x \quad \frac{dy}{dx} = -\csc^2 x$$

Example 15Differentiate **a** $y = x \tan 2x$ **b** $y = \tan^4 x$ u^4

a) $\frac{dy}{dx} = 1 \times \tan 2x + x (2 \sec^2 2x)$

b) $\frac{dy}{dx} = 4 \tan^3 x \times \sec^2 x$

$$= \tan 2x + 2x \sec^2 2x$$

Example 16Show that if $y = \operatorname{cosec} x$, then $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$.

$$y = \frac{1}{\sin x} \quad \text{quotient rule}$$

$$\frac{dy}{dx} = \frac{0 - 1 \times \cos x}{\sin^2 x} = -\frac{\cos x}{\sin^2 x}$$

$$= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x$$