

Differentiation

9.1 Differentiating $\sin x$ and $\cos x$

1 a $6 \cos x$ b $-\frac{3}{2} \cos \frac{1}{2}x$ c $\frac{5}{3} \cos 5x$

d $-4 \sin 4x$ e $-30 \sin 6x$ f $\frac{7}{2} \sin 7x$

2 a $4 \cos x + 3 \sin x$ b $\cos 2x - 4 \sin x - \frac{3}{2} \cos \frac{1}{4}x$

c $\frac{15x^{\frac{1}{2}}}{2} + \frac{8}{x^3} - 2 \cos \frac{1}{3}x$ d $-\frac{4}{x^{\frac{3}{2}}} - \frac{1}{2} \sin \frac{1}{6}x$

3 $x = -59.0^\circ, 121.0^\circ$

4 $y = 3x - \frac{\pi}{2}$

5 $(\frac{\pi}{2}, 9), (\frac{7\pi}{6}, -\frac{9}{2}), (\frac{3\pi}{2}, -3), (\frac{11\pi}{6}, -\frac{9}{2})$

6 $y = -\frac{1}{12}x + \frac{\pi}{12} - 4$

7 $h(\frac{\pi}{2}) = \frac{24 + \pi^2}{8}$; $h'(x) = 3 \cos x + 4 \sin x + x$; $h'(\frac{\pi}{2}) = \frac{8 + \pi}{2}$

Then use the formula $y - h(\frac{\pi}{2}) = h'(\frac{\pi}{2})(x - \frac{\pi}{2})$ to find the equation of the tangent.

8
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h}$$
$$= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$
$$= \lim_{h \rightarrow 0} \left(\left(\frac{\cos h - 1}{h} \right) \cos x - \left(\frac{\sin h}{h} \right) \sin x \right)$$

Since $\frac{\cos h - 1}{h} \rightarrow 0$ and $\frac{\sin h}{h} \rightarrow 1$ the expression inside the limit tends to $(0 \times \cos x - 1 \times \sin x)$

So $\lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = -\sin x$

Hence the derivative of $\cos x$ is $-\sin x$.

9.2 Differentiating exponentials and logarithms

1 a $-18e^{-3x}$ b $10e^{2x} + 7e^{-x}$

c $-\frac{35}{e^{3x}}$ d $2e^{2x} - 2e^{-2x}$

2 a $\frac{1}{x}$ b $\frac{2}{x}$ c $-\frac{1}{x}$ d $\frac{3}{x}$

3 a $3^x \ln 3$ b $5^{2x} (2 \ln 5)$

c $(\frac{2}{5})^{4x} (4 \ln \frac{2}{5})$ d $-\frac{\ln 8}{8^x}$

4 $y = x \ln 2 + \frac{1}{2}(1 + \ln 2)$

5 $-e^{-x} + \frac{6}{x}$

6 $x = 1, y = 3e, \frac{dy}{dx} = 3e - 2$

Rearrange $y - 3e = -\frac{1}{3e-2}(x-1)$ to obtain

$x + (3e-2)y - 9e^2 + 6e - 1 = 0$

7 $x = \frac{1}{2}, y = \frac{5}{2}, \frac{dy}{dx} = \frac{1}{2} \ln \frac{1}{4} + 2 \ln 4 = \ln 8$

Rearrange $y - \frac{5}{2} = (\ln 8)(x - \frac{1}{2})$ to obtain

$y = (\ln 8)x + \frac{5}{2} - \frac{1}{2} \ln 8$

8 a $f(x) = 9^{3x} + 3^{6x} = (3^2)^{3x} + 3^{6x} = 3^{6x} + 3^{6x} = 2(3^{6x})$

b $f'(x) = 2(3^{6x} \times 6 \times \ln 3) = (4 \ln 3)(3 \times 3^{6x}) = (4 \ln 3)(3^{6x+1})$

9.3 The chain rule

1 a $-9(4-3x)^2$ b $\frac{5}{2}(6+5x)^{-\frac{1}{2}}$

c $24x^2(5+2x^3)^3$ d $-30x(7+3x^2)^{-6}$

e $\frac{15}{(4-5x)^{\frac{3}{2}}}$ f $\frac{1}{(6-x)^2}$

g $\frac{8}{3}x^3(3+2x^4)^{-\frac{2}{3}}$ h $\frac{21}{2}x^2(7x^3-4)^{-\frac{1}{2}}$

2 a $3 \sin^2 x \cos x$ b $\frac{4}{x}$

c $(-4 \sin 4x)e^{\cos 4x}$ d $\frac{2 \ln x}{x}$

e $-\tan x$ f $-3e^{-3x} \cos(e^{-3x})$

3 a $-\frac{2}{x}(3 - \ln 6x)$ b $-\frac{12}{e^2}$

4 $\frac{1}{20}$

5 $x + 10y - 12 = 0$

6 $9x - 16y - 30 = 0$

7 $Q(-\frac{5\pi}{6}, -\frac{3}{4}), \frac{dy}{dx} = 10 \sin x \cos x$

Simplify $y + \frac{3}{4} = \frac{5\sqrt{3}}{2}(x + \frac{5\pi}{6})$ to obtain

$12y - 30\sqrt{3}x - 25\pi\sqrt{3} + 9 = 0$

8 $A(-\sqrt{3}, \frac{2\pi}{3}), \frac{dy}{dx} = -\frac{1}{8 \cos 4y}$

Simplify $y - \frac{2\pi}{3} = -4(x + \sqrt{3})$ to obtain $4x + y + 4\sqrt{3} - \frac{2\pi}{3} = 0$

9 $(8 \ln 2)x + y + 8 \ln 2 - 2 = 0$

9.4 The product rule

1 a $(4-5x)^3 - 15x(4-5x)^2$

b $9x^2(2x-1)^6 + 36x^3(2x-1)^5$

c $\frac{96}{x^2}(6x-7)^3 - \frac{8}{x^3}(6x-7)^4$

d $16x(2x^3-4)^3 - 144x^4(2x^3-4)^2$

2 a $2x \cos 2x - 2x^2 \sin 2x$

b $3e^{3x} \sin 5x + 5e^{3x} \cos 5x$

c $-e^{-x}(2x^2+3)^3 + 12xe^{-x}(2x^2+3)^2$

d $-6 \cos 4x \cos 2x + 12 \sin 4x \sin 2x$

e $x + 2x \ln x$

f $4e^{4x} \ln(\cos 3x) + 3e^{4x} \tan 3x$

3 76

4 a $-2xe^{-2x}(x^3-2x^2-3x+3)$

b $12e^{3x} \cos 2x - 8e^{3x} \sin 2x - 3e^{3x} \sin x - e^{3x} \cos x$

c $2x \ln 4x + x - \frac{5}{x} \cos \frac{1}{2}x + \frac{5}{2}(\ln 4x) \sin \frac{1}{2}x$

5 a $-e^{-x} \cos 3x - 3e^{-x} \sin 3x$ b $4x^3 \ln(\sin x) + x^4 \cot x$

6 a $\frac{dy}{dx} = x^3(2-5x^2)^2(-100x^2+16)$

$n=2, a=-100, b=0$ and $c=16$

b $x=0, \pm\sqrt{\frac{2}{5}}, \pm\frac{2}{5}$

7 a $f'(x) = e^{4x}(4 \sin x + \cos x) = 0$

$e^{4x} \neq 0$, so $4 \sin x + \cos x = 0 \Rightarrow \tan x = -\frac{1}{4}$

b $f(\frac{\pi}{2}) = e^{2\pi}, f'(\frac{\pi}{2}) = 4e^{2\pi}$

Therefore the gradient of the normal is $-\frac{1}{4e^{2\pi}}$

Simplify $y - e^{2\pi} = -\frac{1}{4e^{2\pi}}(x - \frac{\pi}{2})$ to obtain

$x + 4e^{2\pi}y - 4e^{4\pi} - \frac{\pi}{2} = 0$

8 $\frac{dy}{dx} = \sin 2x + 2(x-\pi) \cos 2x$

At $x = \frac{3\pi}{4}, y = \frac{\pi}{4}$ and $\frac{dy}{dx} = -1$

Simplify $y - \frac{\pi}{4} = -1(x - \frac{3\pi}{4})$ to obtain $x + y = \pi$

9.5 The quotient rule

$$1 \quad a \quad \frac{42}{(2x-9)^2} \quad b \quad \frac{6x(6-5x)}{(3-5x)^2}$$

$$c \quad \frac{-8(3x^3+2)^2+18x^2(8x-1)(3x^3+2)}{(3x^3+2)^4}$$

$$d \quad \frac{7x^3+84x-54}{(x^2+6)^{\frac{3}{2}}}$$

$$2 \quad a \quad \frac{8x(\cos 2x + x \sin 2x)}{\cos^2 2x}$$

$$b \quad \frac{\ln x + e^{-x} \ln x + \frac{1}{x} e^{-x} - 1}{(\ln x)^2}$$

$$c \quad \frac{e^x(\cos x + 2 \sin x)}{\cos^3 x}$$

$$d \quad \frac{e^{-x} \sin 4x - 4(5 - e^{-x}) \cos 4x}{\sin^2 4x}$$

$$3 \quad -\frac{34}{9}$$

$$4 \quad a \quad \frac{-2x^2+14x-6}{e^{2x}} \quad b \quad y + \frac{5}{e^2} = -\frac{e^2}{6}(x-1)$$

$$5 \quad \frac{2 \sin x(\cos x - \sin x)}{e^{2x}}$$

$$6 \quad \frac{e^{4x}(4x^2-22x+5)}{x^2(x-5)^2}$$

$$7 \quad \sqrt{2}e^{\frac{\pi}{8}}x + 3y - 1 - \frac{3\pi}{8} = 0$$

$$8 \quad a \quad f'(x) = \frac{-2e^x \cos x \sin x - e^x \cos^2 x}{e^{2x}}$$

$$f'(x) = 0 \Rightarrow -2e^x \cos x \sin x - e^x \cos^2 x = 0$$

$$\text{Therefore } 2\cos x \sin x + \cos^2 x = 0$$

$$\text{Solve to obtain } \cos x = 0 \text{ or } \tan x = -\frac{1}{2}$$

$$b \quad A: x = -1.57; B: x = -0.46; C: x = 1.57; D: x = 2.68$$

9.6 Differentiating trigonometric functions

$$1 \quad a \quad 4 \sec^2 4x \quad b \quad -14 \operatorname{cosec}^2 2x$$

$$c \quad -5 \cot^4 x \operatorname{cosec}^2 x \quad d \quad -6 \tan x \sec^2 x$$

$$2 \quad a \quad -6 \operatorname{cosec} 6x \cot 6x \quad b \quad -20 \sec 5x \tan 5x$$

$$c \quad -\frac{4}{3}(\operatorname{cosec} x)^{\frac{4}{3}} \cot x \quad d \quad -24 \sec^3 4x \tan 4x$$

$$3 \quad a \quad 2x \tan 3x + 3x^2 \sec^2 3x$$

$$b \quad \frac{(x^2+1) \sec x \tan x e^{\sec x} - 2x e^{\sec x}}{(x^2+1)^2}$$

$$c \quad -2 \cot x$$

$$d \quad -e^{-x} \cot 4x - 4e^{-x} \operatorname{cosec}^2 4x$$

$$e \quad \frac{\sec^2 x(2x \ln x \tan x - 1)}{x(\ln x)^2}$$

$$f \quad \frac{12(4x-1)^2 \cot^2 x + 2(4x-1)^3 \cot x \operatorname{cosec}^2 x}{\cot^4 x}$$

$$4 \quad a \quad 2 \cos 2y \quad b \quad \frac{1}{2\sqrt{1-x^2}}$$

$$5 \quad y = \operatorname{cosec} x = \frac{1}{\sin x}$$

$$u = 1, \frac{du}{dx} = 0; v = \sin x, \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\sin x \times 0 - 1 \times \cos x}{\sin^2 x} = -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x$$

$$6 \quad a \quad \text{Use the chain rule to find } f'(x) = \frac{-2 \operatorname{cosec}^2 2x}{\cot 2x} \text{ and then}$$

$$\text{use trigonometric identities to obtain } f'(x) = -4 \operatorname{cosec} 4x$$

$$b \quad y = \frac{1}{4}x - \frac{\pi}{32}$$

$$7 \quad \frac{5}{4}$$

$$8 \quad a \quad 4 \sec 4y \tan 4y \quad b \quad \frac{1}{4x\sqrt{x^2-1}}$$

9.7 Parametric differentiation

$$1 \quad a \quad \frac{3t^2}{2} \quad b \quad \frac{t^4}{2} \quad c \quad \frac{e^{4t} + 4te^{4t}}{3}$$

$$d \quad \frac{5}{4} \sec^3 t \quad e \quad \frac{1}{6e^{2t}} \quad f \quad \frac{1}{6t^3}$$

$$2 \quad a \quad \sin t \quad b \quad y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}$$

$$3 \quad a \quad -\left(\frac{3t+4}{2}\right) \quad b \quad 2x + 5y + 61 = 0$$

$$4 \quad a \quad 4 \cot 2\theta \quad b \quad (1, 4) \text{ and } (1, -4)$$

$$5 \quad a \quad -4 \cos^3 \theta \quad b \quad y = 2x - 7$$

$$6 \quad y = 2x + 28$$

$$7 \quad a \quad \frac{\operatorname{cosec}^2 t}{2 \sec 2t \tan 2t}$$

$$b \quad m = -\frac{\sqrt{3}}{3}, \text{ simplify } y + 2\sqrt{3} = -\frac{\sqrt{3}}{3}(x+4)$$

$$\text{to obtain } x\sqrt{3} + 3y + 10\sqrt{3} = 0$$

$$8 \quad a \quad y = -\frac{25}{36}x + \frac{307}{540}$$

$$b \quad \text{Rearrange parametric equation for } x \text{ to } t = \frac{x}{x+1}$$

$$\text{Then } y = \frac{1}{1 + \left(\frac{x}{x+1}\right)^2} = \frac{(x+1)^2}{(x+1)^2 + x^2} = \frac{x^2 + 2x + 1}{2x^2 + 2x + 1}$$

9.8 Implicit differentiation

$$1 \quad u = y^5, \frac{du}{dy} = 5y^4$$

$$\frac{d}{dx}(y^5) = \frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = 5y^4 \frac{dy}{dx}$$

$$2 \quad \frac{d}{dx}(x^3 y^2) = x^3 \frac{d}{dx}(y^2) + \frac{d}{dx}(x^3) y^2$$

$$= x^3 \times 2y \frac{dy}{dx} + 3x^2 y^2$$

$$= 2x^3 y \frac{dy}{dx} + 3x^2 y^2$$

$$3 \quad a \quad \frac{4x^3}{3y^2} \quad b \quad \frac{6x - 10xy}{5x^2 - 8y}$$

$$c \quad \frac{2xy}{10y + 3y^2 - x^2} \quad d \quad \frac{4y}{3(x-5)}$$

$$e \quad \frac{\sin x}{2 \sin y \cos y} \quad f \quad \frac{2ye^{2x} + e^{-y}}{xe^{-y} - e^{2x}}$$

$$4 \quad a \quad \frac{8xy - 2x}{2y - 4x^2} \quad b \quad \frac{10}{9}$$

$$c \quad y = \frac{10}{9}x - \frac{29}{9}$$

$$5 \quad \frac{7}{13}$$

$$6 \quad \frac{2e^{-2x} - 4y}{4x + 3e^{-3y}}$$

$$7 \quad a \quad \frac{\sin 2x}{\cos 2y} \quad b \quad x + 2y = 0$$

$$8 \quad x + y + \frac{\pi}{12} = 0$$

9.9 Using second derivatives

$$1 \quad a \quad -e^{-x} \cos 2x - 2e^{-x} \sin 2x \quad b \quad e^{-x}(4 \sin 2x - 3 \cos 2x)$$

$$2 \quad a \quad 16 + 48e^{4x}$$

$$b \quad 16 \geq 0 \text{ and } 48e^{4x} \geq 0 \text{ for all values of } x, \text{ hence } f(x) \text{ is}$$

$$\text{convex for all values of } x.$$

$$3 \quad a \quad -4 \cos 2x$$

$$b \quad -\frac{\pi}{2} \leq x \leq -\frac{\pi}{4} \text{ and } \frac{\pi}{4} \leq x \leq \frac{\pi}{2}$$

$$4 \quad a \quad f'(x) = (2x-1)^5 + 10x(2x-1)^4$$

$$f''(x) = 20(2x-1)^3(6x-1)$$

$$b \quad f''\left(\frac{1}{2}\right) = 20 \times 0^3 \times 2 = 0$$

$$f''\left(\frac{1}{6}\right) = 20 \times \left(-\frac{2}{3}\right)^3 \times 0 = 0$$

$$c \quad f''(0.1) = 4.096 > 0 \quad f''(0.2) = -0.864 < 0$$

As there is a sign change, this confirms there is a point of inflection at $x = \frac{1}{6}$

- 5 $x \geq \frac{8}{3}$
- 6 a $f'(x) = (2x - 5)^4 + 8x(2x - 5)^3$
 $f''(x) = 16(2x - 5)^3 + 48x(2x - 5)^2$
- b $f''(1) = 0$ and $f''(0.95) = -15.376 < 0$ and
 $f''(1.05) = 13.456 > 0$
 So there is a point of inflection at $x = 1$
- 7 $0.55 \leq x \leq \frac{\pi}{4}$
- 8 $x \leq -3 - \sqrt{3}$ and $-3 + \sqrt{3} \leq x \leq 0$

9.10 Rates of change

- 1 a $8x$ b 24
- 2 $\frac{3}{4\sqrt{2}}(5\pi - 4)$
- 3 $-\frac{3e^8}{64}$
- 4 a $4\pi r^2$ b $12\pi r^2$
- 5 a $\frac{dP}{dt} = kP$
 b The value of the constant k would be negative, or
 $\frac{dP}{dt} = -kP$
- 6 $\frac{dI}{dt} = -k\sqrt{I}$
- 7 a $\pounds 4083.69$ b Decreasing by $\pounds 482.25$ per year
- 8 a Substitute $r = \left(\frac{S}{4\pi}\right)^{\frac{1}{3}}$ into $V = \frac{4}{3}\pi r^3$ and simplify.
 b $\frac{dV}{dS} = \frac{1}{4}\sqrt{\frac{S}{\pi}}$, $\frac{dV}{dt} = \frac{dV}{dS} \times \frac{dS}{dt} = \frac{1}{4}\sqrt{\frac{S}{\pi}} \times 4 = \sqrt{\frac{S}{\pi}}$

Problem solving: Set A

- B a $\frac{dy}{dx} = 4x^3 - 20x^2 + 28x + 6$
 $\frac{d^2y}{dx^2} = 12x^2 - 40x + 28$
- b $\frac{d^2y}{dx^2} = 12x^2 - 40x + 28 = 4(x - 1)(3x - 7)$
 For $1 \leq x \leq \frac{7}{3}$, $x - 1 \geq 0$ and $3x - 7 \leq 0$
 Hence $\frac{d^2y}{dx^2} \leq 0$ and the curve is concave.
- S $\frac{d^2y}{dx^2} = 24x \ln x^2 + 40x = 8x(3 \ln x^2 + 5)$
 When $x > 0$, $8x > 0$. So solve $3 \ln x^2 + 5 \geq 0$, to get $x \geq e^{-\frac{5}{6}}$
 Hence $\frac{d^2y}{dx^2} \geq 0$ when $x \geq e^{-\frac{5}{6}}$ and the curve is convex.
- G $a \geq \frac{1}{2}$

Problem solving: Set B

- B a $\frac{2x + y - 1}{1 - x}$ b $y = -x + 2$
- S a $(x + 1)^2 + (y - 3)^2 = 4$; centre $(-1, 3)$ and radius 2
 b $2 \sin t - 1 = 0 \Rightarrow t = \frac{\pi}{6}, \frac{5\pi}{6}$
 $y = 3 + \sqrt{3}$ or $y = 3 - \sqrt{3}$, therefore $y = 3 + \sqrt{3}$ at B
 c $3 + 2\sqrt{3}$
- G $y = x + \frac{\pi}{6}$ and $y = x - \frac{7\pi}{6}$