

06-10

1 Differentiate with respect to  $x$ :

a  $\ln x^2$

(3 marks)

b  $x^2 \sin 3x$

(4 marks)

a)  $u = x^2$        $y = \ln u$        $\frac{dy}{du} = \frac{1}{u} = \frac{1}{x^2}$   
 $\frac{du}{dx} = 2x$   
 $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{x^2} \times 2x \Rightarrow \frac{2}{x}$

b)  $\frac{dy}{dx} = 2x \sin 3x + x^2 (3 \cos 3x)$

3 Differentiate, with respect to  $x$ :

a  $\frac{\sin x}{x}, x > 0$

(4 marks)

b  $\ln \left( \frac{1}{x^2 + 9} \right)$

(4 marks)

a)  $\frac{dy}{dx} = \frac{x \cos x - \sin x}{x^2}$

b)  $\frac{1}{x^2 + 9} \left( \frac{1}{x^2 + 9} \right)' = \cancel{(x^2 + 9)} \left( \frac{0 - (2x)}{(x^2 + 9)^2} \right)$   
 $= \frac{-2x}{x^2 + 9}$

6 Given that a curve has equation  $y = \cos^2 x + \sin x$ ,  $0 < x < 2\pi$ , find the coordinates of the stationary points of the curve.

(6 marks)

$y = \cos^2 x$        $u = \cos x$        $y = u^2 \Rightarrow \frac{dy}{du} = 2u = 2 \cos x$   
 $\downarrow$   
 $\frac{du}{dx} = -\sin x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -2 \cos x \sin x$$

$$\frac{dy}{dx} = -2 \cos x \sin x + \cos x = 0$$

$$\cos x (1 - 2 \sin x) = 0 \quad \cos x = 0 \quad x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$1 - 2 \sin x = 0 \quad \sin x = \frac{1}{2} \quad x = \frac{\pi}{6}, \frac{5\pi}{6}$$

9  $f(x) = e^{2x} \sin 2x$ ,  $0 < x < \pi$

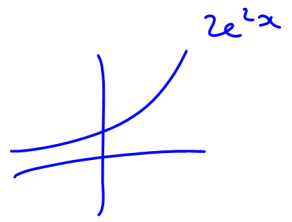
a Use calculus to find the coordinates of the turning points on the graph of  $y = f(x)$ . (6 marks)

b Show that  $f''(x) = 8e^{2x} \cos 2x$ . (4 marks)

c Hence, or otherwise, determine which turning point is a maximum and which is a minimum. (3 marks)

d Find the points of inflection of  $f(x)$ . (2 marks)

a)  $f(x) = 2e^{2x} \sin 2x + 2e^{2x} \cos 2x = 0$   
 $\frac{2e^{2x} (\sin 2x + \cos 2x) = 0}{2e^{2x} \neq 0}$   
 $\sin 2x + \cos 2x = 0 \quad \div \cos 2x$



$\tan 2x + 1 = 0 \quad \tan 2x = -1 \quad 2x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

$x = \frac{3\pi}{8}, \frac{7\pi}{8}$

$f'(x) = 4e^{2x} (\sin 2x + \cos 2x) + 2e^{2x} (2\cos 2x - 2\sin 2x)$

$= 4e^{2x} (\cancel{\sin 2x} + \cos 2x + \cancel{\cos 2x} - \cancel{\sin 2x})$

$= 4e^{2x} (2\cos 2x) \Rightarrow f''(x) = 8e^{2x} \cos 2x$

d) points of inflection  $f''(x) = 0$

$8e^{2x} \neq 0 \quad \cos 2x = 0 \quad 2x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$

c) from (b)  $x = \frac{3\pi}{8}, \frac{7\pi}{8}$

$f''(x) = 8e^{2x} \cos 2x \quad 8e^{2x} > 0 \quad \cos(2\pi/8)$

plug  $x = \frac{3\pi}{8}$  if  $f''(x) > 0$  min  
 $f''(x) < 0$  max

13 Given that  $x = y^2 \ln y$ ,  $y > 0$ ,

a find  $\frac{dx}{dy}$

(4 marks)

b Use your answer to part a to find in terms of e, the value of  $\frac{dy}{dx}$  at  $y = e$ .

(2 marks)

a)  $\frac{dx}{dy} = 2y \ln y + \frac{y^2}{y}$

$\frac{dx}{dy} = 2y \ln y + y$

$\frac{dy}{dx} = \frac{1}{2y \ln y}$

$y = e \quad \frac{dy}{dx} = \frac{1}{2e \ln e}$

15 The diagram shows part of the curve with equation

$y = f(x)$  where  $f(x) = x(1+x) \ln x$ ,  $x > 0$

The point A is the minimum point of the curve.

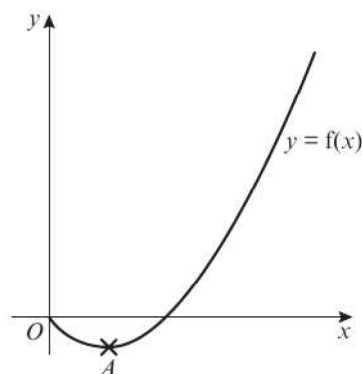
a Find  $f'(x)$ .

(4 marks)

b Hence show that the x-coordinate of A is

the solution to the equation  $x = e^{-\frac{1+x}{1+2x}}$

(4 marks)



a)  $f(x) = x \ln x + x^2 \ln x$

$f'(x) = \ln x + 1 + 2x \ln x + x$

b) At A  $\frac{dy}{dx} = 0$

$\ln x + 1 + 2x \ln x + x = 0$

$\ln x + 2x \ln x = -1 - x$

$\ln x (1+2x) = -(1+x) \Rightarrow \ln x = \frac{-(1+x)}{1+2x}$

$x = e^{-\frac{1+x}{1+2x}}$

16 The curve  $C$  is given by the equations

$$x = 4t - 3, \quad y = \frac{8}{t^2}, \quad t > 0$$

where  $t$  is a parameter.

At  $A$ ,  $t = 2$ . The line  $l$  is the normal to  $C$  at  $A$ .

a Find  $\frac{dy}{dx}$  in terms of  $t$ .

b Hence find an equation of  $l$ .

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$a) \frac{dx}{dt} = 4$$

$$y = 8t^2$$

$$\frac{dy}{dt} = -16t^{-1} = -\frac{16}{t}$$

$$\frac{dy}{dx} = -\frac{16}{t} \times \frac{1}{4} = -\frac{4}{t}$$

$$b) \quad t = 2 \quad \frac{dy}{dx} = -\frac{4}{2} = -2 \quad \text{normal } m = 1/2$$

$$t = 2 \quad x = 4 \times 2 - 3 = 5$$

$$(5, 2)$$

$$y = \frac{8}{2^2} = 2$$

$$l: y - 2 = \frac{1}{2}(x - 5)$$

$$2y - 4 = x - 5 \Rightarrow x - 2y - 1 = 0$$

19 A curve  $C$  is given by the equations

$$x = 2 \cos t + \sin 2t, \quad y = \cos t - 2 \sin 2t, \quad 0 < t < \pi$$

where  $t$  is a parameter.

a Find  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$  in terms of  $t$ .

(3 marks)

$$\frac{dx}{dt} = -2 \sin t + 2 \cos 2t$$

$$\frac{dy}{dt} = -\sin t - 4 \cos 2t$$

b Find the value of  $\frac{dy}{dx}$  at the point  $P$  on  $C$  where  $t = \frac{\pi}{4}$

(3 marks)

$$\frac{dy}{dx} = \frac{-\sin t - 4 \cos 2t}{-2 \sin t + 2 \cos 2t} \quad t = \frac{\pi}{4}$$

24 A circular patch of oil on the surface of some water has radius  $r$  and the radius increases over time at a rate inversely proportional to the radius.

Write down a differential equation relating  $r$  and  $t$ , where  $t$  is the time which has elapsed since the start of the observation.

$$\frac{dr}{dt} \propto \frac{1}{r} \quad \frac{dr}{dt} = \frac{k}{r}$$

$$\int r \, dr = \int k \, dt$$

$$\frac{1}{2} r^2 = kt + c$$

27 The diagram shows the curve  $C$  with parametric equations

$$x = a \sin^2 t, \quad y = a \cos t, \quad 0 \leq t \leq \frac{1}{2}\pi$$

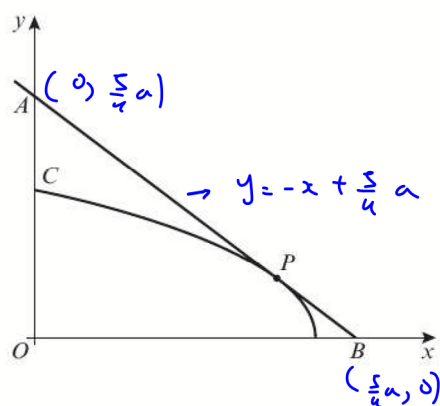
where  $a$  is a positive constant. The point  $P$  lies on  $C$  and has coordinates  $(\frac{3}{4}a, \frac{1}{2}a)$ .

a Find  $\frac{dy}{dx}$ , giving your answer in terms of  $t$ . (4 marks)

b Find an equation of the tangent to  $C$  at  $P$ . (4 marks)

The tangent to  $C$  at  $P$  cuts the coordinate axes at points  $A$  and  $B$ .

c Show that the triangle  $AOB$  has area  $ka^2$  where  $k$  is a constant to be found. (2 marks)



a)  $\frac{dx}{dt} = 2a \sin t \cos t$   $\frac{dy}{dt} = -a \sin t$

$$\frac{dy}{dx} = \frac{-a \sin t}{2a \sin t \cos t} \Rightarrow \frac{dy}{dx} = -\frac{1}{2 \cos t}$$

b)  $P \left( \frac{3}{4}a, \frac{1}{2}a \right)$   $x = a \sin^2 t, \quad y = a \cos t, \quad 0 \leq t \leq \frac{1}{2}\pi$

$$\frac{3}{4}a = a \sin^2 t \quad \sin t = \pm \frac{\sqrt{3}}{2}$$

$$\sin t = \sqrt{3}/2 \Rightarrow t = \pi/3$$

$$\sin t = -\sqrt{3}/2 \Rightarrow t = -\pi/3$$

$$\frac{dy}{dx} = -\frac{1}{2 \cos t} \quad t = \pi/3$$

$$= -\frac{1}{2 \cos(\pi/3)} = -1 \quad m = -1 \quad \left( \frac{3}{4}a, \frac{1}{2}a \right)$$

$$y - \frac{1}{2}a = -1 \left( x - \frac{3}{4}a \right)$$

$$y - \frac{1}{2}a = -x + \frac{1}{4}a \quad y = -x + \frac{5}{4}a$$

$$A = \frac{1}{2} \left( \frac{5}{4}a \right)^2 = \frac{25}{32}a^2$$

29 Find the gradient of the curve with equation  $5x^2 + 5y^2 - 6xy = 13$  at the point (1, 2). (7 marks)

$$10x + 10y \frac{dy}{dx} - 6y - 6x \frac{dy}{dx} = 0$$

$$x=1, y=2$$

$$10 + 10y \frac{dy}{dx} - 12 - 6 \frac{dy}{dx} = 0 \Rightarrow 14 \frac{dy}{dx} - 2 = 0$$

$$\frac{dy}{dx} = \frac{1}{7}$$

30 Given that  $e^{2x} + e^{2y} = xy$ , find  $\frac{dy}{dx}$  in terms of  $x$  and  $y$ .

(7 marks)

$$2e^{2x} + 2e^{2y} \frac{dy}{dx} = y + x \frac{dy}{dx}$$

$$2e^{2y} \frac{dy}{dx} - x \frac{dy}{dx} = y - 2e^{2x}$$

$$\frac{dy}{dx} = \frac{y - 2e^{2x}}{2e^{2y} - x}$$

32 a If  $(1+x)(2+y) = x^2 + y^2$ , find  $\frac{dy}{dx}$  in terms of  $x$  and  $y$ .

(4 marks)

$$(1+y) + (1+x) \frac{dy}{dx} = 2x + 2y \frac{dy}{dx}$$

$$\frac{dy}{dx} (1+x-2y) = 2x-1-y \quad \frac{dy}{dx} = \frac{2x-1-y}{1+x-2y}$$

b Find the gradient of the curve  $(1+x)(2+y) = x^2 + y^2$  at each of the two points where the curve meets the  $y$ -axis.  $\rightarrow x=0$

(3 marks)

$$(1+0)(2+y) = 0 + y^2$$

$$2+y = y^2$$

$$y^2 - y - 2 = 0$$

$$(y-2)(y+1) = 0$$

$$y=2, \quad y=-1$$

$$A(0, 2) \quad B(0, -1)$$

$$A: \quad \frac{dy}{dx} = \frac{0-1-2}{1+0-4} = \frac{-3}{-3} = 1$$

$$B: \quad \frac{dy}{dx} = \frac{0-1+1}{1+0+1} = \frac{0}{2} = 0$$

c Show also that there are two points at which the tangents to this curve are parallel to the  $y$ -axis.

(4 marks)

$\rightarrow \infty \rightarrow \frac{dy}{dx} = \frac{\infty}{0} \rightarrow \text{denominator must be zero}$

$$\frac{dy}{dx} = \frac{2x-2y}{1+x-2y}$$

$$1+x-2y=0$$

$$2y = 1+x$$

$$y = \frac{1}{2} + \frac{1}{2}x$$

$$(1+x)(2+y) = x^2 + y^2$$

$$(1+x)\left(2 + \frac{1}{2} + \frac{1}{2}x\right) = x^2 + \left(\frac{1}{2} + \frac{1}{2}x\right)^2$$

$$(1+x)\left(\frac{5}{2} + \frac{1}{2}x\right) = x^2 + \frac{1}{4} + \frac{1}{2}x + \frac{1}{4}x^2 \quad \text{solve}$$