

10 Trigonometric identities and equations

10.1 Angles in all four quadrants

- 1 Use the unit circle to determine the quadrants where the following trigonometric functions are positive.

- a $\sin \theta$
- b $\cos \theta$
- c $\tan \theta$
- d all of $\sin \theta$, $\cos \theta$ and $\tan \theta$

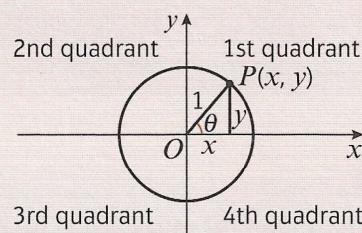
Hint $P(x, y)$ moves on a unit circle such that OP makes an angle θ with the positive x axis.

$$\cos \theta = x = x\text{-coordinate of } P$$

$$\sin \theta = y = y\text{-coordinate of } P$$

$$\tan \theta = \frac{y}{x} = \text{gradient of } OP$$

This means that the sign, (+ or -), of the trigonometric functions depends on the sign of x and y .



- 2 Identify the quadrant that θ lies in as OP moves around the unit circle and hence state whether the trigonometric function of θ is positive or negative.

- a $\theta = 116^\circ$, $\cos \theta$
- b $\theta = 240^\circ$, $\tan \theta$
- c $\theta = 420^\circ$, $\sin \theta$
- d $\theta = -100^\circ$, $\tan \theta$
- e $\theta = 350^\circ$, $\sin \theta$
- f $\theta = -80^\circ$, $\cos \theta$

- 3 Without using a calculator, write down the values of:

- a $\cos 90^\circ$
- b $\cos(-180^\circ)$
- c $\sin 540^\circ$
- d $\sin(-90^\circ)$
- e $\tan 135^\circ$
- f $\tan(-45^\circ)$
- g $\sin 225^\circ$
- h $\cos(-45^\circ)$
- i $\tan(-90^\circ)$

Hint When OP makes an angle of 45° with the positive x -axis the triangle formed is isosceles.

$$\cos 45^\circ = x = \frac{1}{\sqrt{2}}$$

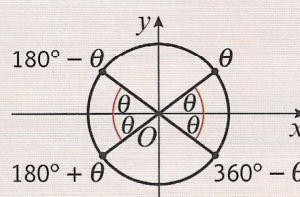
$$\sin 45^\circ = y = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{y}{x} = 1$$

- 4 Given that θ is an acute angle, express in terms of $\sin \theta$, $\cos \theta$ or $\tan \theta$:

- a $\sin(180^\circ + \theta)$
- b $\cos(360^\circ - \theta)$
- c $\tan(180^\circ - \theta)$
- d $\sin(-\theta)$
- e $\tan(180^\circ + \theta)$
- f $\cos(\theta - 180^\circ)$
- g $\sin(\theta - 360^\circ)$
- h $\cos(540^\circ - \theta)$
- i $\tan(\theta - 180^\circ)$

Hint



- P 5** Express in terms of trigonometric ratios of acute angles:

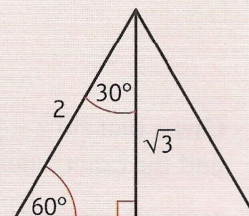
a $\sin 220^\circ$ **b** $\cos 160^\circ$ **c** $\tan 210^\circ$
d $\sin(-100^\circ)$ **e** $\tan 130^\circ$ **f** $\cos 280^\circ$
g $\sin 95^\circ$ **h** $\cos(-200^\circ)$ **i** $\tan(-115^\circ)$

10.2 Exact values of trigonometric ratios

- 1** Express the following as trigonometric ratios of either 30° , 45° or 60° and hence state the exact value.

a $\sin 150^\circ$ **b** $\sin(-45^\circ)$
c $\sin 315^\circ$ **d** $\sin 240^\circ$
e $\cos 150^\circ$ **f** $\cos(-60^\circ)$
g $\cos(-135^\circ)$ **h** $\cos 120^\circ$
i $\tan 225^\circ$ **j** $\tan(-150^\circ)$
k $\tan 405^\circ$ **l** $\tan(-120^\circ)$

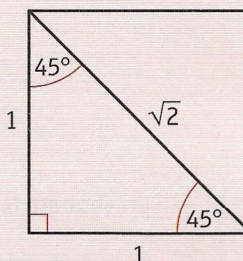
Hint The exact values of the trigonometric ratios of 30° and 60° can be worked out using Pythagoras' theorem by considering an equilateral triangle of side 2 units.



$$\sin 30^\circ = \frac{1}{2} \quad \cos 30^\circ = \frac{\sqrt{3}}{2} \quad \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2} \quad \cos 60^\circ = \frac{1}{2} \quad \tan 60^\circ = \sqrt{3}$$

The exact values of the trigonometric ratios of 45° can be worked out by considering a square of side 1 unit.



$$\sin 45^\circ = \frac{1}{\sqrt{2}} \quad \cos 45^\circ = \frac{1}{\sqrt{2}} \quad \tan 45^\circ = 1$$

- 2** The table shows some of the values of x and y that satisfy the equation $y = a \sin x + b$.

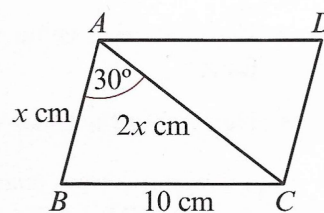
x	0°	30°	60°	90°	120°	150°	180°
y	3	4	$3 + \sqrt{3}$	5	$3 + \sqrt{3}$	4	3

Find the value of y when $x = 45^\circ$.

- 3** The diagram shows parallelogram $ABCD$, where $AB = DC = x$ cm and $AD = BC = 10$ cm.

Given that $AC = 2x$ cm and $\angle BAC = 30^\circ$, show that

$$x^2 = \frac{100(5 + 2\sqrt{3})}{13}$$



10.3 Trigonometric identities

- 1** Write down the value of:

a $2 \sin^2 \theta - (1 - \cos^2 \theta) + \cos^2 \theta$
b $6 \sin^2 \frac{\theta}{2} + 6 \cos^2 \frac{\theta}{2}$ **c** $(1 + \cos x)^2 + (1 - \cos x)^2 + 2 \sin^2 x$

Hint Use the trigonometric identity $\sin^2 \theta + \cos^2 \theta = 1$.

2 Write in terms of $\tan x$:

a $\frac{2 \sin x}{\cos x}$ b $\frac{3 \sin x}{\sqrt{1 - \sin^2 x}}$ c $\frac{1 - \cos^2 x}{1 - \sin^2 x}$

3 Prove that $\frac{\sin^2 x - \cos^2 x}{1 - \sin^2 x} \equiv \tan^2 x - 1$

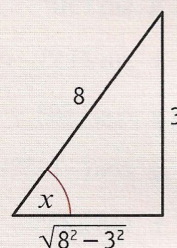
Hint Use the trigonometric identity $\tan x \equiv \frac{\sin x}{\cos x}$ for all values of x such that $\cos x \neq 0$

Hint To prove an identity, start from the LHS and manipulate the expression until it matches the RHS. You can quote the standard identities in your proof. ← Sections 7.4, 7.5

4 Given that $\sin x = \frac{3}{8}$ and that x is obtuse, find the exact value of:

a $\cos x$ b $\tan x$

Hint You can use $\cos^2 x \equiv 1 - \sin^2 x$ to work out the exact value of $\cos x$, then $\tan x \equiv \frac{\sin x}{\cos x}$. Remember you need to check the sign of cosine and tangent in the required quadrant. You can check your answer using basic trigonometry and Pythagoras' theorem in a right-angled triangle.



E 5 a Given that $\sin \theta = 6 \cos \theta$, find the value of $\tan \theta$. (1 mark)

b Express $4 \cos^2 \theta - 5 \sin^2 \theta$ in terms of $\sin \theta$. (1 mark)

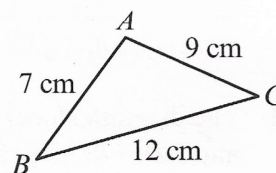
E/P 6 Show that the equation $4 \cos^2 x - 3 \sin^2 x = 2$ can be written as $7 \cos^2 x = 5$. (2 marks)

E/P 7 Prove that $\frac{\cos^2 x - \sin^2 x}{\cos^2 x - \sin x \cos x} \equiv 1 + \tan x$ (3 marks)

E/P 8 The diagram shows the triangle ABC with $AB = 7$ cm, $BC = 12$ cm and $AC = 9$ cm.

a Find the exact value of $\cos B$, giving your answer in its simplest form. (3 marks)

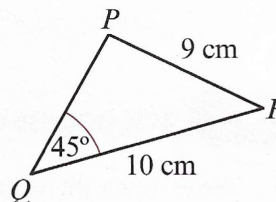
b Hence find the exact value of $\sin B$. (2 marks)



E/P 9 The diagram shows triangle PQR with $PR = 9$ cm, $QR = 10$ cm and angle $PQR = 45^\circ$.

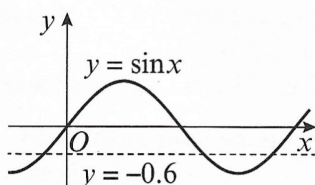
a Show that $\sin P = \frac{5\sqrt{2}}{9}$ (3 marks)

b Given that angle P is obtuse, find the exact value of $\cos P$. (2 marks)



10.4 Simple trigonometric equations

- 1 The diagram shows a sketch of $y = \sin x$.



- a Use your calculator to find the principal solution to the equation $\sin x = -0.6$.
- b Use the graph and your answer to part a to find solutions to the equation $\sin x = -0.6$ in the interval $0 \leq x \leq 360^\circ$.

Hint Use the inverse trigonometric function on your calculator to get the **principal value**.

Your calculator will give the principal values in these ranges:

$$\cos^{-1}: 0 \leq x \leq 180^\circ$$

$$\sin^{-1}: -90^\circ \leq x \leq 90^\circ$$

$$\tan^{-1}: -90^\circ < x < 90^\circ$$

- 2 Find the solutions to the equation $\cos \theta = \frac{1}{\sqrt{2}}$ in the interval $0 \leq \theta \leq 360^\circ$.

Hint Find the first solution using your calculator or your knowledge of exact values of trigonometric functions. Draw the graph of $y = \cos \theta$ for the given interval and use the properties of the graph to find the second solution.

← Section 9.5

- 3 Solve the equation $4 \sin x = 3$ in the interval $0 \leq x \leq 360^\circ$.

Hint First rewrite the equation in the form $\sin x = \dots$

- 4 Solve the equation $\sin \theta = -10 \cos \theta$ in the interval $0 < \theta \leq 360^\circ$.

Hint You can use the identity $\tan \theta = \frac{\sin \theta}{\cos \theta}$ to solve trigonometric equations. The principal value of $\tan \theta$ is not in the given interval. As the period for the graph of $y = \tan \theta$ is 180° , you can find successive solutions by adding or subtracting multiples of 180° .

- E** 5 A student's attempt to solve the equation $5 \tan x = 9 \sin x$ in the interval $-90^\circ < x < 90^\circ$ is shown.

- a Identify two mistakes made by this student, giving a brief explanation of each mistake. (2 marks)
- b Write down the correct solutions to the original equation. (1 mark)

$$5 \tan x = 9 \sin x$$

$$5 \frac{\sin x}{\cos x} = 9 \sin x$$

$$5 \sin x = 9 \sin x \cos x$$

$$5 = 9 \cos x$$

$$\cos x = \frac{5}{9}$$

$$x = 56.3^\circ \text{ (to 3 s.f.)}$$

- E/P 6** a Given that $3 \sin \theta = 5 \cos \theta$, find the value of $\tan \theta$. (1 mark)
 b Hence, or otherwise, find the values of θ in the interval $0 \leq x \leq 360^\circ$ for which $3 \sin \theta = 5 \cos \theta$, giving your answers to 1 decimal place. (3 marks)
- E/P 7** Find all the values of x for which $\cos^2 x = \frac{3}{4}$ in the interval $-180^\circ \leq x < 180^\circ$. (5 marks)
- E/P 8** a Show that the equation $4 \cos^2 x - 3 \sin^2 x = 1$ can be written as $7 \cos^2 x = 4$. (2 marks)
 b Hence solve the equation $4 \cos^2 x - 3 \sin^2 x = 1$ for $0 \leq x \leq 360^\circ$, giving your answers to 1 decimal place. (7 marks)
- E/P 9** a Solve $(2 + \tan \theta)(4 \sin \theta - 1) = 0$ for $-180^\circ \leq \theta < 180^\circ$. (4 marks)
 b Solve $\frac{6 \sin x}{\tan x} = 1$ for $0 \leq x < 360^\circ$. (6 marks)

10.5 Harder trigonometric equations

- 1 Find the values of θ in the interval $0 \leq \theta \leq 360^\circ$ for which:

a $\cos 3\theta = \frac{\sqrt{3}}{2}$ b $\sin \frac{\theta}{2} = \frac{1}{\sqrt{2}}$
 c $\tan 2\theta = -\sqrt{3}$

Hint In part a, let $X = 3\theta$. If $0 \leq \theta \leq 360^\circ$, then $0 \leq 3\theta \leq 1080^\circ$.

Use the graph of $y = \cos X$ to solve $\cos X = \frac{\sqrt{3}}{2}$ for $0 \leq X \leq 1080^\circ$.

There will be six solutions. Divide the solutions for X by 3 to find the solutions for θ .

- 2 Solve the equations in the given intervals:

a $\sin(\theta + 30^\circ) = 0.2$, $0 \leq \theta \leq 360^\circ$
 b $\cos(60^\circ - \theta) = -0.4$, $-180^\circ \leq \theta \leq 180^\circ$
 c $\tan(\theta - 20^\circ) = 9$, $0 \leq \theta \leq 360^\circ$

Hint In part a, let $X = \theta + 30^\circ$ and solve $\sin X = 0.2$ for $30^\circ \leq X \leq 390^\circ$.

Subtract 30° from each of the solutions for X to find solutions for θ .

- 3 Solve the equations in the given intervals:

a $\cos 3\theta = \sqrt{3} \sin 3\theta$, $-180^\circ \leq \theta \leq 180^\circ$ b $2 \sin 4\theta - 5 \cos 4\theta = 0$, $0 \leq \theta \leq 360^\circ$
 c $5 \sin(\theta + 40^\circ) + 8 \cos(\theta + 40^\circ) = 0$, $-180^\circ \leq \theta \leq 180^\circ$

- 4 Solve the equations in the given intervals:

a $\sin(2x - 30^\circ) = 0.75$, $0 \leq x \leq 360^\circ$
 b $\cos(3x + 20^\circ) = -0.4$, $0 \leq x \leq 180^\circ$
 c $\tan(4x - 45^\circ) = 2$, $0 \leq x \leq 180^\circ$
 d $2 \sin(2x + 10^\circ) = 3 \cos(2x + 10^\circ)$, $-180^\circ \leq x \leq 180^\circ$

Hint In part a, let $X = 2x - 30^\circ$. The end points in the interval for X are $2 \times 0 - 30^\circ = -30^\circ$ and $2 \times 360^\circ - 30^\circ = 690^\circ$. Find the solutions for X , then use $2x - 30^\circ = X$ for each solution to find the solutions for x .

- E 5** Solve, for $0 \leq x < 360^\circ$,
- a $\cos 3x = -\frac{1}{2}$ (6 marks)
- b $\sin(x - 20^\circ) = -\frac{1}{\sqrt{2}}$ (4 marks)
- E 6** a Solve $5 \sin 2x = 3 \cos 2x$ for $0 \leq x \leq 360^\circ$, giving your answers to 1 decimal place. (5 marks)
- b Find the solutions of the equation $\sin(3x - 15^\circ) = \frac{1}{\sqrt{2}}$ for which $0 \leq x \leq 180^\circ$. (6 marks)
- E 7** a Find all values of θ , to 1 decimal place, in the interval $0 \leq \theta \leq 360^\circ$ for which $5 \sin(\theta + 30^\circ) = 4$. (4 marks)
- b Find all values of θ , to the nearest degree, in the interval $0 \leq \theta \leq 360^\circ$ for which $\tan^2 \frac{\theta}{2} = 16$. (5 marks)
- E 8** Solve $\tan(5x - 20^\circ) = \frac{5}{2}$ for $-90^\circ < x < 90^\circ$, giving answers to 1 decimal place. (6 marks)
- E 9** a Show that the equation $\tan 2x = 4 \sin 2x$ can be written in the form $(1 - 4 \cos 2x) \sin 2x = 0$ (2 marks)
- b Hence solve $\tan 2x = 4 \sin 2x$ for $0 \leq x \leq 180^\circ$, giving your answers to 1 decimal place where appropriate. You must show clearly how you obtained your answers. (5 marks)

10.6 Equations and identities

- 1** Solve the equations for $0 \leq \theta \leq 360^\circ$.
Give your answers to 3 significant figures where they are not exact.

a $9 \sin^2 \theta = 1$ b $4 \cos^2 \theta - 1 = 0$

c $5 \sin^2 \theta - \sin \theta = 0$ d $\tan^2 \theta + \tan \theta - 12 = 0$

e $(3 \cos \theta - 2)^2 = 1$ f $\tan^2 3\theta = 4$

Hint You need to be able to solve quadratic equations in trigonometric functions, which can lead to two sets of solutions.

← Section 2.1

- 2** Solve the equations for $0 \leq \theta \leq 360^\circ$.
Give your answers to 3 significant figures where they are not exact.

a $\cos^2 3\theta = 1$ b $\tan^2 \theta = 5 \tan \theta$

c $\sin^2 \theta - 3 \sin \theta - 4 = 0$ d $4 \sin(\theta + 30^\circ) = \tan(\theta + 30^\circ)$

Hint There are no valid solutions to $\sin x = k$ or $\cos x = k$ when $k < -1$ or $k > 1$.

- 3** a Show that the equation $2 \cos^2 x - \sin^2 x - 3 \cos x + 2 = 0$ can be written as $3 \cos^2 x - 4 \cos x + 1 = 0$

Hint You can use $\sin^2 x = 1 - \cos^2 x$ to convert to a quadratic equation in $\cos x$.

- b Hence solve the equation $2 \cos^2 x - \sin^2 x - 3 \cos x + 2 = 0$ for $0 \leq x \leq 360^\circ$.

- 4 Solve the equations for $0 \leq \theta \leq 360^\circ$.
Give your answers to 1 decimal place where they are not exact.

a $5 \cos^2 \theta = 6(1 - \sin \theta)$

b $8 - 7 \cos \theta = 6 \sin^2 \theta$

c $3 \tan \theta = 2 \cos \theta$

Hint You can use $\cos^2 \theta \equiv 1 - \sin^2 \theta$ and $\tan \theta \equiv \frac{\sin \theta}{\cos \theta}$ to convert to a quadratic equation.

- E/P** 5 a Show that the equation $2 \sin^2 x - 3 \cos^2 x = 1$ can be written as $5 \sin^2 x = 4$. (2 marks)
b Hence solve the equation $2 \sin^2 x - 3 \cos^2 x = 1$ for $0 \leq x < 360^\circ$, giving your answers to one decimal place. (7 marks)
- E/P** 6 Solve $2 \cos^2 x + \sin^2 x = \frac{10}{9}$ for $0 < x < 180^\circ$, giving your answers to one decimal place. (4 marks)
- E/P** 7 Find all the solutions of $2 \cos^2 x + 2 = 7 \sin x$ for $0 \leq x \leq 360^\circ$. (6 marks)
- 8 a Show that the equation $\sin \theta \tan \theta = 2 \cos \theta + 3$ can be written in the form $3 \cos^2 \theta + 3 \cos \theta - 1 = 0$ (3 marks)
b Hence solve $\sin \theta \tan \theta = 2 \cos \theta + 3$ for $0 \leq \theta < 360^\circ$, showing each stage of your working. (5 marks)
- E/P** 9 Solve $15 \sin^2 x + 8 \cos x - 16 = 0$ for $360^\circ \leq x < 540^\circ$. Give your answers to one decimal place. (5 marks)

Problem solving Set A

Bronze

- a Given that $2 \sin x = 5 \cos x$, write down the value of $\tan x$. (1 mark)
b Hence solve $2 \sin(\theta + 20^\circ) = 5 \cos(\theta + 20^\circ)$ for $0 \leq \theta < 360^\circ$, giving your answers to one decimal place. (5 marks)

Silver

Solve the equation $\frac{\sin 3x + 1}{1 - \sin 3x} = 2$, for $0 \leq x \leq 180^\circ$.

Give your answers to one decimal place.

(6 marks)

Gold

Solve the equation $4 \cos(2x - 15^\circ) + 3 = 0$, for $-180^\circ \leq x \leq 180^\circ$.

Give your answers to one decimal place.

(7 marks)

Problem solving Set B

Bronze

- a Show that the equation $\frac{3 \sin^2 x + \cos^2 x}{\cos^2 x} = 5$ can be written as $\tan^2 x = \frac{4}{3}$ (2 marks)
- b Hence solve the equation $\tan^2 x = \frac{4}{3}$ for $0 \leq x < 360^\circ$, giving your answers to one decimal place. (6 marks)

Silver

- a Show that the equation $4 \cos^2 x + 9 \sin x - 6 = 0$ can be written as $4 \sin^2 x - 9 \sin x + 2 = 0$. (2 marks)
- b Hence solve $4 \cos^2 x + 9 \sin x - 6 = 0$ for $0 \leq x < 720^\circ$, giving your answers to one decimal place (6 marks)

Gold

Solve $5(\cos 3x - 1) = 3 \sin 3x \tan 3x$, for $0 \leq x < 90^\circ$, giving your answers, where appropriate, to one decimal place. (9 marks)

Now try this → Exam question bank Q7, Q47, Q60, Q66, Q76, Q84, Q85