

$$\begin{array}{c}
 1 \\
 1 \quad 1 \\
 1 \quad 2 \quad 1 \\
 1 \quad 3 \quad 3 \quad 1 \\
 1 \quad 4 \quad 6 \quad 4 \quad 1
 \end{array}$$

$(a+b)^0 = 1$
 $(a+b)^1 = 1a + 1b$
 $(a+b)^2 = 1a^2 + 2ab + 1b^2$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Example 2

The coefficient of x^2 in the expansion of $(2 - cx)^3$ is 294. Find the possible value(s) of the constant c .

$$\begin{aligned}
 (2 - cx)^3 &= 2^3 + 3 \times 2^2(-cx) + 3 \times 2(-cx)^2 + (-cx)^3 \\
 &= 8 - 12cx + 6c^2x^2 - c^3x^3
 \end{aligned}$$

$$6c^2 = 294 \quad c = \pm \sqrt{\frac{294}{6}} \quad c = \pm 7$$

7 The coefficient of x^3 in the expansion of $(2 - x)(3 + bx)^3$ is 45. Find possible values of the constant b .

$$\begin{aligned}
 (2 - x)(3^3 + 3 \times 3^2bx + 3 \times 3b^2x^2 + b^3x^3) \\
 (2 - x)(27 + 27bx + 9b^2x^2 + b^3x^3) \\
 2b^3x^3 - 9b^2x^3 = (2b^3 - 9b^2)x^3 \\
 2b^3 - 9b^2 = 45 \quad 2b^3 - 6b^2 - 45 = 0
 \end{aligned}$$

$$\begin{aligned}
 5! &= 5 \times 4 \times 3 \times 2 \times 1 \\
 &= 120
 \end{aligned}$$

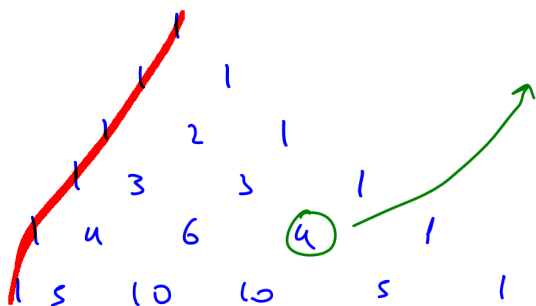
$$n! = n(n-1)(n-2)!$$

$${}^nC_r \text{ or } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$${}^5C_2 = \frac{5!}{2!(5-2)!} = \frac{5 \times 4 \times \cancel{3!}}{2 \times \cancel{3!}} = 10$$

${}^5C_2$ 5th row in Pascal's triangle
2nd number

We don't measure lot row and
lot number



$${}^4C_3 = \frac{4!}{3!(4-3)!} = \frac{4 \times \cancel{3!}}{\cancel{3!} \times 1} = 4$$

${}^5C_2$

9 Show that:

a ${}^nC_1 = n$

b ${}^nC_2 = \frac{n(n-1)}{2}$

a) ${}^nC_1 = \frac{n!}{1!(n-1)!} = \frac{n \times \cancel{(n-1)!}}{1 \times \cancel{(n-1)!}} = n$

b) ${}^nC_2 = \frac{n!}{2!(n-2)!} = \frac{n(n-1) \times \cancel{(n-2)!}}{2 \times \cancel{(n-2)!}} = \frac{n(n-1)}{2}$

Example 4

Use the binomial theorem to find the expansion of $(3 - 2x)^5$.

$$(3 - 2x)^5 = 3^5 + \binom{5}{1} 3^4 (-2x) + \binom{5}{2} 3^3 (-2x)^2 + \binom{5}{3} 3^2 (-2x)^3 + \binom{5}{4} 3 (-2x)^4 + (-2x)^5$$

$$= 243 - 810x + 1080x^2 - 720x^3 + 240x^4 - 32x^5$$

7 Find the binomial expansion of $(x + \frac{1}{x})^5$ giving each term in its simplest form. (4 marks)

$$= x^5 + \binom{5}{1} x^4 \left(\frac{1}{x}\right) + \binom{5}{2} x^3 \left(\frac{1}{x}\right)^2 + \binom{5}{3} x^2 \left(\frac{1}{x}\right)^3 + \binom{5}{4} x \left(\frac{1}{x}\right)^4 + \binom{5}{5} \left(\frac{1}{x}\right)^5$$

$$= x^5 + 5x^3 + 10x + 10x^{-1} + 5x^{-3} + x^{-5}$$

Example 6

a Find the coefficient of x^4 in the binomial expansion of $(2 + 3x)^{10}$.

$$\binom{10}{4} 2^6 (3x)^4 = 1,080,640x^4$$

b Find the coefficient of x^3 in the binomial expansion of $(2+x)(3-2x)^7$.

$$\begin{matrix} \downarrow & \downarrow \\ x^2 & x^1 \end{matrix}$$

$$(2+x) \left(\dots + \binom{7}{5} 3^5 (-2x)^2 + \binom{7}{3} 3^4 (-2x)^3 + \dots \right)$$

$$(2+x) (\dots + 20,412x^2 - 27,680x^3 + \dots)$$

$$20,412x^3 - 45,360x^3 = -24,948x^3$$

Example 7

$g(x) = (1+kx)^{10}$, where k is a constant.
Given that the coefficient of x^3 in the binomial expansion of $g(x)$ is 15, find the value of k .

$$\binom{10}{3} (kx)^3 = 120k^3x^3 \quad 120k^3 = 15$$

$$k = \sqrt[3]{\frac{15}{120}} = \frac{1}{2}$$

Example 8

- a Write down the first three terms, in ascending powers of x , of the binomial expansion of $(1+qx)^8$, where q is a non-zero constant.
- b Given that, in the expansion of $(1+qx)^8$, the coefficient of x is $-r$ and the coefficient of x^2 is $7r$, find the value of q and the value of r .

$$\begin{aligned} a) (1+qx)^8 &= 1 + \binom{8}{1} qx + \binom{8}{2} q^2 x^2 + \dots \\ &= 1 + 8qx + 28q^2x^2 + \dots \end{aligned}$$

$$\begin{aligned} b) \quad 8q &= -r & 28q^2 &= 7r & \frac{28q^2}{8q} &= \frac{7r}{-r} \\ & & & & \frac{7}{2}q &= -7 \Rightarrow q = -2 \\ & & & & r &= -8q \\ & & & & r &= 16 \end{aligned}$$

- 6 a Find the first three terms, in ascending powers of x , of the binomial expansion of $(5+px)^{30}$, where p is a non-zero constant. (2 marks)
- b Given that in this expansion the coefficient of x^2 is 29 times the coefficient of x work out the value of p . (4 marks)

$$\begin{aligned} a) (5+px)^{30} &= 5^{30} + \binom{30}{1} 5^{29} px + \binom{30}{2} 5^{28} p^2 x^2 + \dots \\ &= 5^{30} + 30 \times 5^{29} px + 435 \times 5^{28} p^2 x^2 + \dots \end{aligned}$$

$$\begin{aligned} b) \quad 435 \times 5^{28} p^2 &= 29 \times 30 \times 5^{29} p \\ 435p &= 29 \times 30 \\ p &= 2 \end{aligned}$$

9 a Write down the first three terms, in ascending powers of x , of the binomial expansion of $(1 + px)^{15}$, where p is a non-zero constant. (2 marks)

b Given that, in the expansion of $(1 + px)^{15}$, the coefficient of x is $(-q)$ and the coefficient of x^2 is $5q$, find the value of p and the value of q . (4 marks)

$$a) (1+px)^{15} = 1 + \binom{15}{1} px + \binom{15}{2} p^2 x^2 + \dots$$

$$= 1 + 15px + 105p^2 x^2 + \dots$$

$$b) 15p = -q \quad \frac{105p^2}{15p} = \frac{5q}{-q} \Rightarrow 7p = -5 \quad p = -5/7$$

$$q = -15p \\ = -15 \times (-5/7) = \frac{75}{7}$$

Example 9

a Find the first four terms of the binomial expansion, in ascending powers of x , of $(1 - \frac{x}{4})^{10}$.

b Use your expansion to estimate the value of 0.975^{10} , giving your answer to 4 decimal places.

$$a) (1 - \frac{x}{4})^{10} = 1^{10} + \binom{10}{1} 1^9 (-\frac{x}{4}) + \binom{10}{2} 1^8 (-\frac{x}{4})^2 + \binom{10}{3} 1^7 (-\frac{x}{4})^3 + \dots$$

$$= 1 - 2.5x + \frac{45}{16} x^2 - \frac{15}{8} x^3 + \dots$$

$$b) 0.975^{10} = (1 - \frac{x}{4})^{10} \quad 0.975 = 1 - \frac{x}{4}$$

$$x = (1 - 0.975) \times 4 \Rightarrow x = 0.10$$

$$0.975^{10} \approx 1 - 2.5 \text{ Ans} + \frac{45}{16} \text{ Ans}^2 - \frac{15}{8} \text{ Ans}^3$$

$$0.975^{10} \approx 0.7763 \text{ (4 d.p.)}$$

In calculator $0.975^{10} = 0.77633$

7 a Find the first 3 terms, in ascending powers of x , of the binomial expansion of $(3 - \frac{x}{5})^{10}$, giving each term in its simplest form. (4 marks)

b Explain how you would use your expansion to give an estimate for the value of 2.98^{10} . (1 mark)

$$a) (3 - \frac{x}{5})^{10} = 3^{10} + \binom{10}{1} 3^9 (-\frac{x}{5}) + \binom{10}{2} 3^8 (-\frac{x}{5})^2 + \dots$$

$$= 59,049 - 39,366x + 11,801.8x^2 + \dots$$

$$b) 3 - \frac{x}{5} = 2.98 \Rightarrow x = (3 - 0.98) \times 5 \quad x = 0.1$$

$$2.98^{10} \approx 59,049 - 39,366 \text{ Ans} + 11,801.8 \text{ Ans}^2 \approx 55,250.5$$

9 The coefficient of x^2 in the binomial expansion of $\left(1 + \frac{x}{2}\right)^n$, where n is a positive integer, is 7.

a Find the value of n .

(2 marks)

b Using the value of n found in part a, find the coefficient of x^4 .

(4 marks)

$$\begin{aligned} a) \quad \binom{n}{2} \left(\frac{x}{2}\right)^2 &= \frac{n!}{2!(n-2)!} \cdot \frac{1}{4} x^2 \\ &= \frac{n(n-1) \cancel{(n-2)!}}{2 \cancel{(n-2)!}} \times \frac{1}{4} x^2 = \frac{n(n-1)}{8} x^2 \end{aligned}$$

$$\begin{aligned} \frac{n(n-1)}{8} &= 7 \Rightarrow n^2 - n = 56 \Rightarrow n^2 - n - 56 = 0 \\ (n-8)(n+7) &= 0 \\ n &= 8 \end{aligned}$$

$$\begin{aligned} b) \quad \left(1 + \frac{x}{2}\right)^8 &= \binom{8}{4} \left(\frac{x}{2}\right)^4 \\ &= {}^8C_4 \times \frac{1}{2^4} x^4 \\ &= \frac{35}{8} x^4 \end{aligned}$$

13 In the binomial expansion of $(2k + x)^n$, where k is a constant and n is a positive integer, the coefficient of x^2 is equal to the coefficient of x^3 .

a Prove that $n = 6k + 2$.

(3 marks)

b Given also that $k = \frac{2}{3}$, expand $(2k + x)^n$ in ascending powers of x up to and including the term in x^3 , giving each coefficient as an exact fraction in its simplest form.

(4 marks)

$$a) \quad \binom{n}{2} (2k)^2 x^2 = \binom{n}{3} (2k)^3 x^3$$

$$\frac{n!}{2!(n-2)!} \frac{(2k)^2}{(2k)^2} = \frac{n!}{3!(n-3)!} \times \frac{(2k)^3}{(2k)^3}$$

$$\frac{n(n-1) \cancel{(n-2)!}}{2 \cancel{(n-2)!}} \times \frac{1}{(2k)^2} = \frac{n(n-1)(n-2) \cancel{(n-3)!}}{6 \cancel{(n-3)!}} \times \frac{1}{(2k)^3}$$

$$\frac{1}{2(2k)^2} = \frac{n-2}{6(2k)^3} \Rightarrow n-2 = \frac{1}{2} \frac{(2k)^3}{(2k)^2}$$

$$n-2 = 6k \Rightarrow n = 6k + 2$$

15 The coefficient of x^2 in the binomial expansion of $(2 + kx)^8$, where k is a positive constant, is 2800.

a Use algebra to calculate the value of k . (2 marks)

b Use your value of k to find the coefficient of x^3 in the expansion. (4 marks)

$$\begin{aligned} a) \quad \binom{8}{2} 2^6 (kx)^2 &= 8 \times 2^6 \times k^2 x^2 \\ &= 1792 k^2 x^2 \end{aligned}$$

$$1792 k^2 = 2800 \rightarrow k = \sqrt{\frac{2800}{1792}} \Rightarrow k = 1.25$$

17 In the binomial expansion of $(2 + px)^5$, where p is a constant, the coefficient of x^3 is 135. Calculate:

a the value of p , (4 marks)

b the value of the coefficient of x^4 in the expansion. (2 marks)

$$\begin{aligned} a) \quad \binom{5}{3} 2^2 (px)^3 &= 40p^3 x^3 & 40p^3 &= 135 & p &= \sqrt[3]{\frac{135}{40}} \\ & & & & p &= 1.5 \end{aligned}$$

21 a Find the first 3 terms, in ascending powers of x , of the binomial expansion of $\left(2 + \frac{x}{2}\right)^7$, giving each term in its simplest form. (4 marks)

b Explain how you would use your expansion to give an estimate for the value of 2.05^7 . (1 mark)

$$a) \quad = 2^7 + \binom{7}{1} 2^6 \left(\frac{x}{2}\right) + \binom{7}{2} 2^5 \left(\frac{x}{2}\right)^2 + \dots$$

$$b) \quad 2 + \frac{x}{2} = 2.05 \rightarrow x = (2.05 - 2) \times 2 = 0.10$$