

07.10

## Quotient Rule

$$y = \frac{f(x)}{g(x)} \quad \frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

7 Find the exact value of  $\frac{dy}{dx}$  at the point  $x = \frac{\pi}{9}$  on the curve with equation  $y = \frac{\ln x}{\sin 3x}$

$$\frac{dy}{dx} = \frac{\frac{1}{x} \sin 3x - \ln x (3 \cos 3x)}{\sin^2 3x}$$

$$x = \frac{\pi}{9}$$

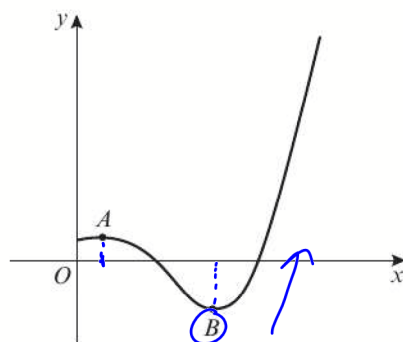
12 The diagram shows a sketch of the curve with equation  $y = f(x)$ , where

$$f(x) = \frac{2 \cos 2x}{e^{2-x}}, \quad 0 < x < \pi$$

The curve has a maximum turning point at  $A$  and a minimum turning point at  $B$  as shown in the diagram.

a Show that the  $x$ -coordinates of point  $A$  and point  $B$  are solutions to the equation  $\tan 2x = \frac{1}{2}$  (4 marks)

b Find the range of  $f(x)$ . (2 marks)



$$a) f'(x) = \frac{-4 \sin 2x (e^{2-x}) - 2 \cos 2x (-e^{2-x})}{(e^{2-x})^2}$$

$$f'(x) = \frac{-4 \sin 2x + 2 \cos 2x}{e^{2-x}} = 0$$

$$-4 \sin 2x + 2 \cos 2x = 0 \quad \div \cos 2x$$

$$-4 \tan 2x + 2 = 0 \Rightarrow \tan 2x = \frac{2}{4} = \frac{1}{2}$$

$$b) \tan 2x = \frac{1}{2} \quad 2x = 0.46, \pi + 0.46$$

$$x = 0.23, 1.80 \quad B$$

$$f(1.80) = f(x) = \frac{2 \cos 2x}{e^{2-x}}$$

$$1.8 \text{ then equal } f(1.8) = \frac{2 \cos(1.8 \text{ rad})}{e^{2-1.8}} = -1.47$$

$$\text{range } f(x) \geq -1.47$$

**Example 14**

If  $y = \tan x$ , find  $\frac{dy}{dx}$

$$y = \frac{\sin x}{\cos x} \quad \frac{dy}{dx} = \frac{\cos x \times \cos x - \sin x (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$y = \tan x \quad \frac{dy}{dx} = \sec^2 x$$

$$\text{eg. } y = \tan 5x \quad \frac{dy}{dx} = 5 \sec^2 5x$$

$$y = \cot x \quad \frac{dy}{dx} = -\operatorname{cosec}^2 x$$

**Example 15**

Differentiate **a**  $y = x \tan 2x$  **b**  $y = \tan^4 x \rightarrow u = \tan x$

$$\text{a) } \frac{dy}{dx} = (1) \tan 2x + x (2 \sec^2 2x) \quad \text{b) } \frac{dy}{dx} = 4 \tan^3 x \sec^2 x$$

$$= \tan 2x + 2x \sec^2 2x$$

**Example 16**

Show that if  $y = \operatorname{cosec} x$ , then  $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$ .

$$y = \frac{1}{\sin x} \quad \frac{dy}{dx} = \frac{0 - 1 \cos x}{\sin^2 x} = -\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x \cot x$$

$$y = \operatorname{cosec} x \quad \frac{dy}{dx} = -\operatorname{cosec} x \cot x$$

$$y = \operatorname{cosec} hx \quad \frac{dy}{dx} = -h \operatorname{cosec} hx \cot hx$$

$$y = \sec x \quad \frac{dy}{dx} = \sec x \tan x$$

**Example 17**

Differentiate: **a**  $y = \frac{\operatorname{cosec} 2x}{x^2}$       **b**  $y = \sec^3 x$

$$a) \frac{dy}{dx} = \frac{-2 \operatorname{cosec} 2x \cot 2x (x^2) - \operatorname{cosec} 2x (2x)}{x^4}$$

$$= \frac{-\operatorname{cosec} 2x (2x \cot 2x + 2)}{x^3}$$

$$b) \frac{dy}{dx} = 3 \sec^2 x (\sec x \tan x) \\ = 3 \sec^3 x \tan x$$

**Example 18**

Show that the derivative of  $\arcsin x$  is  $\frac{1}{\sqrt{1-x^2}}$

$$y = \arcsin x \Rightarrow \sin y = x \Rightarrow \frac{dx}{dy} = \cos y$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

**Example 19**

Given  $y = \arcsin x^2$  find  $\frac{dy}{dx}$

$$y = \arcsin u \quad \frac{dy}{du} = \frac{1}{\sqrt{1+u^2}}$$

$$u = x^2 \quad y = \arcsin u \quad \frac{dy}{du} = \frac{1}{\sqrt{1+u^2}} = \frac{1}{\sqrt{1+x^4}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{\sqrt{1+x^4}} \times 2x = \frac{2x}{\sqrt{1+x^4}}$$

**Example 20**

Given that  $y = \arctan\left(\frac{1-x}{1+x}\right)$ , find  $\frac{dy}{dx}$

$$u = \frac{1-x}{1+x}$$

$$y = \arctan u \quad \frac{dy}{du} = \frac{1}{1+u^2}$$

$$= \frac{1}{1 + \frac{(1-x)^2}{(1+x)^2}}$$

$$\frac{dy}{du} = \frac{1}{\frac{(1+x)^2 + (1-x)^2}{(1+x)^2}} = \frac{(1+x)^2}{1+x^2+x^2+1-x^2+x^2} = \frac{(1+x)^2}{2+2x^2}$$

$$\frac{du}{dx} = \frac{-1(1+x) - (1-x)(1)}{(1+x)^2} = \frac{-1-x-1+x}{(1+x)^2} = \frac{-2}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{\cancel{(1+x)^2}}{2+2x^2} \times \frac{(-2)}{\cancel{(1+x)^2}} = \frac{-2}{2(1+x^2)} = \frac{-1}{1+x^2}$$

10 Given that the curve  $C$  has equation

$$y = \frac{\arctan 2x}{x}$$

a show that the value of  $\frac{dy}{dx}$  when  $x = \frac{\sqrt{3}}{2}$  is  $\frac{3\sqrt{3} - 4\pi}{9}$  (4 marks)

b find the equation of the normal to the curve  $C$  at  $x = \frac{\sqrt{3}}{2}$ . (3 marks)

$$a) \frac{dy}{dx} = \frac{\left(\frac{1}{1+(2x)^2}\right)x - \arctan 2x \times 1}{x^2}$$

$$= \frac{\frac{2x}{1+4x^2} - \arctan 2x}{x^2}$$

$$x = \frac{\sqrt{3}}{2}$$

$$= \frac{\frac{2\sqrt{3}}{1+4\left(\frac{\sqrt{3}}{2}\right)^2} - \arctan\left(x \frac{\sqrt{3}}{x}\right)}{\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{\frac{\sqrt{3}}{1+3} - \frac{\pi}{3}}{\frac{3}{4}} = \frac{\frac{3\sqrt{3} - 4\pi}{4}}{\frac{3}{4}} = \frac{(3\sqrt{3} - 4\pi) \times 4}{3 \cancel{4} \times 3} = \frac{3\sqrt{3} - 4\pi}{1}$$

b) normal:  $m = -\frac{1}{3\sqrt{3} - 4\pi}$  or  $m = \frac{1}{4\pi - 3\sqrt{3}}$

$$x = \frac{\sqrt{3}}{2} \quad y = \frac{\arctan\left(x \frac{\sqrt{3}}{x}\right)}{\frac{\sqrt{3}}{2}} \quad y = \frac{\arctan 2x}{x}$$

$$= \frac{\frac{\pi}{3}}{\frac{\sqrt{3}}{2}} = \frac{2\pi}{3\sqrt{3}} \quad \left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3\sqrt{3}}\right)$$

$$y - \frac{2\pi}{3\sqrt{3}} = \frac{1}{4\pi - 3\sqrt{3}} \left(x - \frac{\sqrt{3}}{2}\right)$$

**Example 21**

Find the gradient at the point  $P$  where  $t = 2$ , on the curve given parametrically by

$$x = t^3 + t, \quad y = t^2 + 1, \quad t \in \mathbb{R}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dt} = 2t$$

$$\frac{dx}{dt} = 3t^2 + 1$$

$$\frac{dt}{dx} = \frac{1}{3t^2 + 1}$$

$$\frac{dy}{dx} = \frac{2t}{3t^2 + 1} \quad t = 2$$

**Example 22**

Find the equation of the normal at the point  $P$  where  $\theta = \frac{\pi}{6}$ , to the curve with parametric equations  $x = 3 \sin \theta$ ,  $y = 5 \cos \theta$ .

$$\frac{dy}{d\theta} = -5 \sin \theta$$

$$\frac{dx}{d\theta} = 3 \cos \theta$$

$$\frac{dy}{dx} = -\frac{5 \sin \theta}{3 \cos \theta} = -\frac{5}{3} \tan \theta$$

$$\frac{dy}{dx} = -\frac{5}{3} \tan \theta \quad \theta = \pi/6$$

$$\frac{dy}{dx} = -\frac{5}{3} \tan(\pi/6) \quad \frac{dy}{dx} = -\frac{5\sqrt{3}}{9} \quad \text{normal} = \frac{9}{5\sqrt{3}}$$

$$\theta = \pi/6 \quad x = 3 \sin(\pi/6) \Rightarrow x = 3/2$$

$$y = 5 \cos(\pi/6) \Rightarrow y = \frac{5\sqrt{3}}{2}$$

$$y - \frac{5\sqrt{3}}{2} = \frac{9}{5\sqrt{3}} \left( x - \frac{3}{2} \right)$$

$$5\sqrt{3}y - \frac{(5\sqrt{3})^2}{2} = 9x - \frac{27}{2}$$

$$5\sqrt{3}y - \frac{75}{2} = 9x - \frac{27}{2} \Rightarrow 9x - 5\sqrt{3}y + 24 = 0$$

7 A curve has parametric equations

$$x = 2 \sin^2 t, \quad y = 2 \cot t, \quad 0 < t < \frac{\pi}{2}$$

a Find an expression for  $\frac{dy}{dx}$  in terms of the parameter  $t$ . (4 marks)

b Find an equation of the tangent to the curve at the point where  $t = \frac{\pi}{6}$ . (4 marks)

$$a) \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \frac{dy}{dt} = -2 \operatorname{cosec}^2 t$$

$$\frac{dx}{dt} = 4 \sin t \times \cos t$$

$$\frac{dy}{dx} = \frac{-2 \operatorname{cosec}^2 t}{4 \sin t \cos t} = \frac{-1}{2 \sin^2 t \cos t}$$

$$\frac{dy}{dx} = \frac{-1}{2 \sin^2 t \cos t} \quad t = \pi/6$$

$$= \frac{-1}{2 (\sin(\pi/6))^2 \cos(\pi/6)} = \frac{-1}{2 (\frac{1}{2})^2 \times \frac{\sqrt{3}}{2}}$$

$$= \frac{-1}{\frac{\sqrt{3}}{8}} = -\frac{8}{\sqrt{3}} = -\frac{8\sqrt{3}}{3}$$

$$x = 2 \sin^2 t, \quad y = 2 \cot t, \quad t = \pi/6$$

$$x = 2 (\sin(\pi/6))^2 \\ = 2 \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

$$y = \frac{2}{\tan t} = \frac{2}{\tan(\pi/6)} = 2\sqrt{3}$$

$$m = -\frac{8\sqrt{3}}{3} \quad \left(\frac{1}{2}, 2\sqrt{3}\right)$$

$$y - 2\sqrt{3} = \frac{-8\sqrt{3}}{3} (x - \frac{1}{2}) \Rightarrow 3y - 6\sqrt{3} = -8\sqrt{3}x + 4\sqrt{3}$$

$$8\sqrt{3}x + 3y - 10\sqrt{3} = 0$$

### Example 23

Find  $\frac{dy}{dx}$  in terms of  $x$  and  $y$  where  $x^3 + x + y^3 + 3y = 6$ .

$$3x^2 + 1 + 3y^2 \frac{dy}{dx} + 3 \frac{dy}{dx} = 0 \quad \frac{dy}{dx} (3y^2 + 3) = -3x^2 - 1$$

$$\frac{dy}{dx} = \frac{-3x^2 - 1}{3y^2 + 3}$$

### Example 24

Given that  $4xy^2 + \frac{6x^2}{y} = 10$ , find the value of  $\frac{dy}{dx}$  at the point (1, 1).

$$4y^2 + 8xy \frac{dy}{dx} + \frac{12xy - 6x^2 \frac{dy}{dx}}{y^2} = 0$$

$$x=1, y=1 \quad 4 + 8 \frac{dy}{dx} + \frac{12 - 6 \frac{dy}{dx}}{1} = 0$$

$$16 + 2 \frac{dy}{dx} = 0 \quad \frac{dy}{dx} = -8$$

Find the value of  $\frac{dy}{dx}$  at the point (1, 1) where  $e^{2x} \ln y = x + y - 2$ .

$$e^{2x} \ln y + \frac{e^{2x}}{y} \frac{dy}{dx} = 1 + 1 \frac{dy}{dx} - 0$$

$$x=1, y=1 \quad \therefore \quad e^2 \cancel{\ln 1} + \frac{e^2}{1} \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} (e^2 - 1) = 1 \quad \frac{dy}{dx} = \frac{1}{e^2 - 1}$$

10 A curve  $C$  satisfies  $\sin x + \cos y = 0.5$ , where  $-\pi < x < \pi$  and  $-\pi < y < \pi$ .

a Find an expression for  $\frac{dy}{dx}$  (2 marks)

b Find the coordinates of the stationary points on  $C$ . (5 marks)

$$a) \cos x - \sin y \frac{dy}{dx} = 0 \quad \frac{dy}{dx} = \frac{\cos x}{\sin y}$$

$$b) \frac{dy}{dx} = 0 \quad \therefore \quad \frac{\cos x}{\sin y} = 0 \quad \Rightarrow \quad \cos x = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}, -\frac{\pi}{2}$$

$$x = -\frac{\pi}{2}, \quad \cos y = 0.5 - \sin x \quad y = \cos^{-1}(0.5 - \sin(-\frac{\pi}{2}))$$

$$x = \frac{\pi}{2} \quad \dots$$

11 The curve  $C$  has the equation  $ye^{-3x} - 3x = y^2$ .

a Find  $\frac{dy}{dx}$  in terms of  $x$  and  $y$ . (5 marks)

b Show that the equation of the tangent to  $C$  at the origin,  $O$ , is  $y = 3x$ . (4 marks)

$$a) e^{-3x} \frac{dy}{dx} + y(-3e^{-3x}) - 3 = 2y \frac{dy}{dx}$$

$$\frac{dy}{dx} (e^{-3x} - 2y) = 3ye^{-3x} + 3$$

$$\frac{dy}{dx} = \frac{3ye^{-3x} + 3}{e^{-3x} - 2y}$$

$$b) x=0, y=0 \quad \frac{dy}{dx} = \frac{0+3}{e^0-0} = \frac{3}{1} = 3$$

$$y-0 = 3(x-0) \quad \Rightarrow \quad y = 3x$$

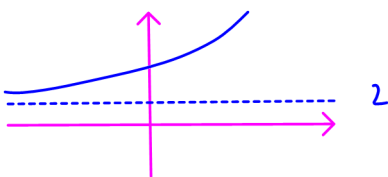
### Example 27

Show that the function  $f(x) = e^{2x} + x^2$  is convex for all real values of  $x$ .

U minimum  $f''(x) > 0$

$$f'(x) = 2e^{2x} + 2x$$

$$f''(x) = 4e^{2x} + 2 > 0 \quad \text{minimum} \\ \therefore \text{convex}$$



### Example 28

The curve  $C$  has equation  $y = x^3 - 2x^2 - 4x + 5$ .

a Show that  $C$  is concave on the interval  $[-2, 0]$  and convex on the interval  $[1, 3]$ .

b Find the coordinates of the point of inflection.

$$a) \frac{dy}{dx} = 3x^2 - 4x - 4 \quad \frac{d^2y}{dx^2} = 6x - 4$$

$$x = -2 \therefore \frac{d^2y}{dx^2} = -16 < 0 \quad \text{concave}$$

$$x = 1 \quad \frac{d^2y}{dx^2} = 2 > 0 \quad \text{convex}$$

$$b) \frac{d^2y}{dx^2} = 0 \quad 6x - 4 = 0 \Rightarrow x = \frac{2}{3}$$

4  $f(x) = 2x^2 \ln x, x > 0$

Show that  $f$  has exactly one point of inflection and determine the value of  $x$  at this point.

$$f(x) = 4x \ln x + 2x^2 \times \frac{1}{x}$$

$$f'(x) = 4 \ln x + 4x \times \frac{1}{x} + 2 \quad f''(x) = 4 \ln x + 6 = 0$$

$$\ln x = -\frac{3}{2} \quad x = e^{-3/2}$$

$$f(e^{-3/2}) = 2(e^{-3/2})^2 \ln e^{-3/2} \quad (e^{-3/2}, -3e^{-9/2})$$

$$= 2e^{-3} \left( -\frac{3}{2} \right) = -3e^{-9/2}$$