

23.10

Example 14

a Solve the equation $\cos 3\theta = 0.766$, in the interval $0 \leq \theta \leq 360^\circ$.

b Solve the equation $2 \sin 2\theta = \cos 2\theta$, in the interval $0 \leq \theta \leq 360^\circ$.

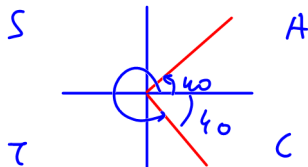
a) $\cos 3\theta = 0.766$ (1st & 4th) interval $0 \leq 3\theta \leq 1080$

$$3\theta = \cos^{-1}(0.766)$$

$$3\theta = 40, 320$$

$$400, 680 \rightarrow +360$$

$$760, 1040 \rightarrow +720$$



$$\theta = \frac{40}{3}, \frac{320}{3}, \frac{400}{3}, \frac{680}{3}, \frac{760}{3}, \frac{1040}{3}$$

b Solve the equation $2 \sin 2\theta = \cos 2\theta$, in the interval $0 \leq \theta \leq 360^\circ$.

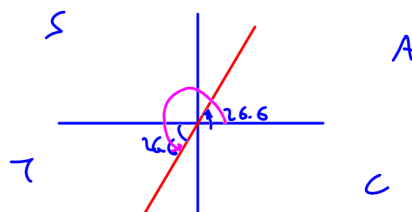
$$0 \leq 2\theta \leq 720$$

$$2 \tan 2\theta = 1$$

$$\tan 2\theta = 1/2 \text{ (1st & 3rd)}$$

$$2\theta = 26.6, 206.6, 388.6, 568.6$$

$$\theta = 13.3, 103.3, 194.3, 284.3$$



Example 16

Solve for θ , in the interval $0 \leq x \leq 360^\circ$, the equations

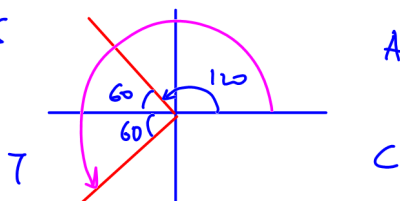
a $2 \cos^2 \theta - \cos \theta - 1 = 0$

b $\sin^2(\theta - 30^\circ) = \frac{1}{2}$

a) $(\cos \theta - 1)(2 \cos \theta + 1) = 0$

$$\cos \theta = 1, \theta = 0, 360$$

$$\cos \theta = -1/2, \theta = 120, 240$$



$$\theta = 0, 120, 240, 360$$

b $\sin^2(\theta - 30^\circ) = \frac{1}{2}$

$$\sin(\theta - 30) = \pm \frac{1}{\sqrt{2}} \begin{cases} \sin(\theta - 30) = 1/\sqrt{2} & (1st \& 2nd) \\ \sin(\theta - 30) = -1/\sqrt{2} & (3rd \& 4th) \end{cases}$$

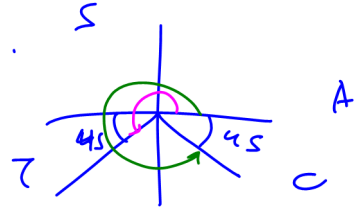
interval $-30 \leq \theta - 30 \leq 330$

$$\sin(\theta - 30) = \frac{1}{\sqrt{2}} \quad \theta - 30 = 45, 135$$

$\theta = 75, 165$

$$\sin(\theta - 30) = -\frac{1}{\sqrt{2}}, \quad \theta - 30 = -45, 129, 315$$

$\theta = 155, 345$



$\theta = 75^\circ, 155^\circ, 165^\circ, 345^\circ$

Example 17

Find the values of x , in the interval $-180^\circ \leq x \leq 180^\circ$, satisfying the equation $2\cos^2 x + 9\sin x = 3\sin^2 x$.

$$\downarrow$$

$$1 - \sin^2 x$$

$$2 - 2\sin^2 x + 9\sin x = 3\sin^2 x$$

$$5\sin^2 x - 9\sin x - 2 = 0$$

$$(\sin x - 2)(5\sin x + 1) = 0$$

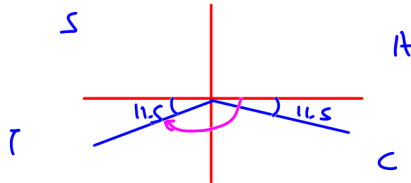
$$\sin x = 2 \Rightarrow \text{no solutions}$$

or

$$\sin x = -\frac{1}{5}, \quad x = -11.5, -180 + 11.5$$

$= -11.5, -168.5$

$$-180^\circ \leq x \leq 180^\circ$$



Example 5

In triangle ABC , $\vec{AB} = \mathbf{a}$ and $\vec{AC} = \mathbf{b}$.

P is the midpoint of AB .

Q divides AC in the ratio $3:2$.

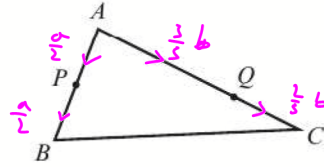
Write in terms of $\mathbf{a}$ and $\mathbf{b}$:

a $\vec{BC}$

b $\vec{AP}$

c $\vec{AQ}$

d $\vec{PQ}$



a) $\vec{BC} = -\mathbf{a} + \mathbf{b}$

b) $\vec{AP} = \frac{1}{2}\mathbf{a}$

c) $\vec{AQ} = \frac{3}{5}\mathbf{b}$

d) $\vec{PQ} = -\frac{1}{2}\mathbf{a} + \frac{1}{5}\mathbf{b}$

11 The vectors $2\mathbf{a} + k\mathbf{b}$ and $5\mathbf{a} + 3\mathbf{b}$ are parallel. Find the value of k .

gradient = $\frac{k}{2} = \frac{3}{5}$ $k = \frac{6}{5}$

or

$2\mathbf{a} + k\mathbf{b} = \lambda(5\mathbf{a} + 3\mathbf{b}) \Rightarrow 2 = 5\lambda$ and $k = 3\lambda$
 $\lambda = \frac{2}{5}$ $k = \frac{6}{5}$

4 Given that $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$, find:

a λ if $\mathbf{a} + \lambda\mathbf{b}$ is parallel to the vector $\mathbf{i}$

b μ if $\mu\mathbf{a} + \mathbf{b}$ is parallel to the vector $\mathbf{j}$

a) $\mathbf{a} + \lambda\mathbf{b} = (2+3\lambda)\mathbf{i} + (5-\lambda)\mathbf{j}$
 parallel to vector $\mathbf{i}$
 means that vector $\mathbf{j} = 0$
 $5-\lambda = 0 \Rightarrow \lambda = 5$

b) $\mu\mathbf{a} + \mathbf{b} = (2\mu+3)\mathbf{i} + (5\mu-1)\mathbf{j}$
 parallel to vector $\mathbf{j}$
 means that vector $\mathbf{i} = 0$
 $2\mu+3 = 0 \Rightarrow \mu = -3/2$

5 Given that $\mathbf{c} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{d} = \mathbf{i} - 2\mathbf{j}$, find:

a λ if $\mathbf{c} + \lambda\mathbf{d}$ is parallel to $\mathbf{i} + \mathbf{j}$

b μ if $\mu\mathbf{c} + \mathbf{d}$ is parallel to $\mathbf{i} + 3\mathbf{j}$

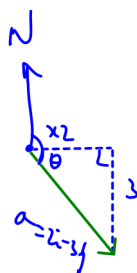
c s if $\mathbf{c} - s\mathbf{d}$ is parallel to $2\mathbf{i} + \mathbf{j}$

d t if $\mathbf{d} - t\mathbf{c}$ is parallel to $-2\mathbf{i} + 3\mathbf{j}$

c) $\mathbf{c} - s\mathbf{d} = (3-s)\mathbf{i} + (4+2s)\mathbf{j}$
 parallel to $2\mathbf{i} + \mathbf{j}$
 $\frac{4+2s}{3-s} = \frac{1}{2}$
 $8+4s = 3-s$
 $5s = -5$
 $s = -1$

d) $\mathbf{d} - t\mathbf{c} = (1-3t)\mathbf{i} + (-2-4t)\mathbf{j}$
 $\frac{3}{-2} = \frac{-2-4t}{1-3t}$
 $3-9t = 4+8t$
 $-1 = 17t \quad t = \frac{1}{17}$

$\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$



magnitude of $\mathbf{a} \rightarrow |\mathbf{a}|$

$|\mathbf{a}| = \sqrt{2^2 + 3^2}$
 $|\mathbf{a}| = \sqrt{13}$

$\tan \theta = 3/2 \quad \theta = \tan^{-1}(3/2)$

bearing $x = 90 + \theta$

unit vector $\hat{a} = \frac{a}{|a|}$

$$\hat{a} = \frac{1}{\sqrt{13}} (2i - 3j)$$

$$\begin{aligned} \hat{a} &= \frac{2}{\sqrt{13}} i - \frac{3}{\sqrt{13}} j & |\hat{a}| &= \sqrt{\left(\frac{2}{\sqrt{13}}\right)^2 + \left(\frac{-3}{\sqrt{13}}\right)^2} \\ & & &= \sqrt{\frac{4}{13} + \frac{9}{13}} \\ & & &= \sqrt{\frac{13}{13}} = 1 \end{aligned}$$

Example 11

Given that $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = -2\mathbf{i} - 4\mathbf{j}$:

a find $|\mathbf{a}|$

b find a unit vector in the direction of $\mathbf{a}$

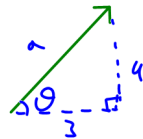
c find the exact value of $|2\mathbf{a} + \mathbf{b}|$

a) $|\mathbf{a}| = \sqrt{3^2 + 4^2} = 5$

b) $\hat{a} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{5} (3\mathbf{i} + 4\mathbf{j})$

$$\hat{a} = 0.6\mathbf{i} + 0.8\mathbf{j}$$

$$\theta = \tan^{-1}(4/3)$$

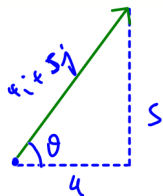


c) $2\mathbf{a} + \mathbf{b} = 4\mathbf{i} + 4\mathbf{j}$

$$|2\mathbf{a} + \mathbf{b}| = \sqrt{4^2 + 4^2} = 4\sqrt{2}$$

Example 12

Find the angle between the vector $4\mathbf{i} + 5\mathbf{j}$ and the positive x -axis.

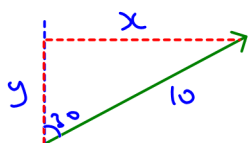


$$\theta = \tan^{-1}(5/4)$$

Example 13

Vector $\mathbf{a}$ has magnitude 10 and makes an angle of 30° with $\mathbf{j}$.

Find $\mathbf{a}$ in $\mathbf{i}, \mathbf{j}$ and column vector format.



$$\mathbf{a} = 5\mathbf{i} + 5\sqrt{3}\mathbf{j}$$

$$\sin 30 = \frac{x}{10}$$

$$x = 10 \sin 30 = 5$$

$$\cos 30 = \frac{y}{10}$$

$$y = 10 \cos 30 = 5\sqrt{3}$$

8 Given that $|2\mathbf{i} - k\mathbf{j}| = 2\sqrt{10}$, find the exact value of k .

$$\sqrt{2^2 + (-k)^2} = 2\sqrt{10}$$

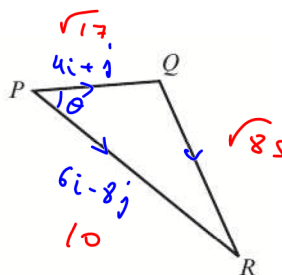
$$4 + k^2 = 40 \Rightarrow k^2 = 36$$

$$k = \pm 6$$

11 In triangle PQR , $\overrightarrow{PQ} = 4\mathbf{i} + \mathbf{j}$, $\overrightarrow{PR} = 6\mathbf{i} - 8\mathbf{j}$.

a Find the size of $\angle QPR$, in degrees, to one decimal place. (5 marks)

b Find the area of triangle PQR . (2 marks)



a) Let: $\vec{QR} = 2\mathbf{i} - 9\mathbf{j}$

$$|\vec{PQ}| = \sqrt{4^2 + 1^2} = \sqrt{17}$$

$$|\vec{PR}| = \sqrt{6^2 + (-8)^2} = 10$$

$$|\vec{QR}| = \sqrt{2^2 + (-9)^2} = \sqrt{85}$$

$$\sqrt{85}^2 = \sqrt{17}^2 + 10^2 - 2\sqrt{17} \times 10 \cos \theta$$

$$85 = 17 + 100 - 20\sqrt{17} \cos \theta$$

$$\cos \theta = \frac{17 - 85}{20\sqrt{17}} \Rightarrow \theta = 67.2^\circ \text{ (1 d.p.)}$$

b) Area: $\frac{1}{2} \times 10\sqrt{17} \sin(67.2)$

Example 14

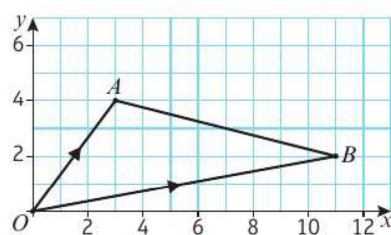
The points A and B in the diagram have coordinates $(3, 4)$ and $(11, 2)$ respectively.

Find, in terms of $\mathbf{i}$ and $\mathbf{j}$:

a the position vector of A

b the position vector of B

c the vector $\overrightarrow{AB}$



a) $\vec{OA} = 3\mathbf{i} + 4\mathbf{j}$ b) $\vec{OB} = 11\mathbf{i} + 2\mathbf{j}$

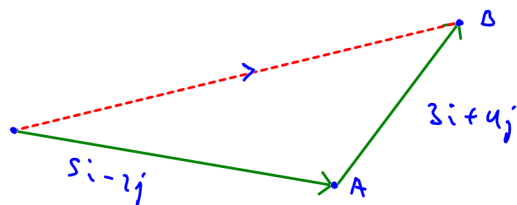
b) $\vec{AB} = -\vec{OA} + \vec{OB}$
 $= 8\mathbf{i} - 2\mathbf{j}$

Example 15

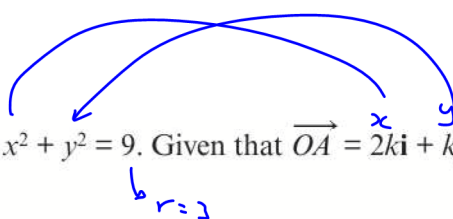
$\vec{OA} = 5\mathbf{i} - 2\mathbf{j}$ and $\vec{AB} = 3\mathbf{i} + 4\mathbf{j}$. Find:

- the position vector of B
- the exact value of $|\vec{OB}|$ in simplified surd form.

$$\vec{OB} = 8\mathbf{i} + 2\mathbf{j}$$



$$\begin{aligned} |\vec{OB}| &= \sqrt{8^2 + 2^2} \\ &= \sqrt{68} \\ &= 2\sqrt{17} \end{aligned}$$



- 7 The point A lies on the circle with equation $x^2 + y^2 = 9$. Given that $\vec{OA} = 2k\mathbf{i} + k\mathbf{j}$, find the exact value of k .

(3 marks)

$$\begin{aligned} |\vec{OA}| &= \sqrt{(2k)^2 + k^2} \\ &= \sqrt{5k^2} = r \end{aligned}$$

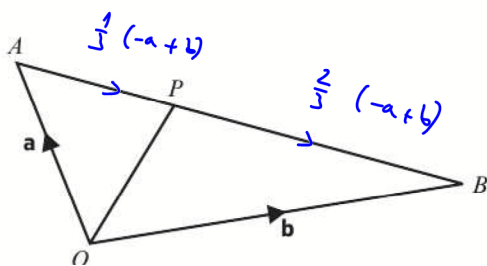
$$\sqrt{5k^2} = 3$$

$$5k^2 = 9 \quad k = \pm \frac{3}{\sqrt{5}}$$

Example 16

In the diagram the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively (referred to the origin O). The point P divides AB in the ratio $1:2$.

Find the position vector of P .



$$\vec{AB} = -\mathbf{a} + \mathbf{b}$$

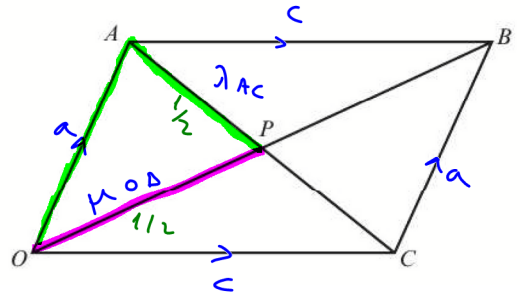
$$\begin{aligned} \vec{OP} &= \mathbf{a} + \frac{1}{3}(-\mathbf{a} + \mathbf{b}) \\ &= \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} \\ &= \frac{1}{3}(2\mathbf{a} + \mathbf{b}) \end{aligned}$$

Example 17

$OABC$ is a parallelogram. P is the point where the diagonals OB and AC intersect.

The vectors $\mathbf{a}$ and $\mathbf{c}$ are equal to $\overrightarrow{OA}$ and $\overrightarrow{OC}$ respectively.

Prove that the diagonals bisect each other.



$$\vec{OP} = \vec{OA} + \vec{AP}$$

$$\vec{AC} = -\mathbf{a} + \mathbf{c}$$

$$\vec{OP} = \mu (\mathbf{a} + \mathbf{c})$$

$$= \mu \mathbf{a} + \mu \mathbf{c}$$

$$= \mathbf{a} + \lambda (-\mathbf{a} + \mathbf{c})$$

$$= \mathbf{a} - \lambda \mathbf{a} + \lambda \mathbf{c}$$

$$= (1 - \lambda) \mathbf{a} + \lambda \mathbf{c}$$

$$1 - \lambda = \mu$$

$$\lambda = \mu$$

$$(1 - \mu = \mu) \Rightarrow \mu = 1/2 \rightarrow \lambda = 1/2$$