

Example 21

The cabin pressure, P , in pounds per square inch (psi) on an aeroplane at cruising altitude can be modelled by the equation $P = 11.5 - 0.5 \sin(t - 2)$, where t is the time in hours since the cruising altitude was first reached, and angles are measured in radians.

- State the maximum and the minimum cabin pressure.
- Find the time after reaching cruising altitude that the cabin first reaches a maximum pressure.
- Calculate the cabin pressure after 5 hours at a cruising altitude.
- Find all the times during the first 8 hours of cruising that the cabin pressure would be exactly 11.3 psi.

a) when $\sin(t-2) = -1 \rightarrow \max P = 11.5 - 0.5(-1) = 12$
 $\sin(t-2) = 1 \rightarrow \min P = 11.5 - 0.5(1) = 11$

b) $\sin(t-2) = -1 \quad t-2 = -\frac{\pi}{2}, \quad t = 2 - \frac{\pi}{2}$

c) $P = 11.5 - 0.5 \sin(5-2)$

d) $11.5 - 0.5 \sin(t-2) = 11.3$ Solve

- 7 a Express $200 \sin \theta - 150 \cos \theta$ in the form $R \sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$
 Give the value of α to 4 decimal places.

(4 marks)

$$R \sin \theta \cos \alpha - R \cos \theta \sin \alpha = 200 \sin \theta - 150 \cos \theta$$

$$R \cos \alpha = 200, \quad R \sin \alpha = 150$$

$$\frac{R \sin \alpha = 150}{R \cos \alpha = 200} \Rightarrow \tan \alpha = \frac{3}{4} \Rightarrow \alpha = 0.6435 \text{ (4 d.p.)}$$

$$R = \sqrt{200^2 + 150^2} \\ = 250$$

- 8 Given that $\sin x \cos y = \frac{1}{2}$ and $\cos x \sin y = \frac{1}{3}$,

- a show that $\sin(x+y) = 5 \sin(x-y)$.

Given also that $\tan y = k$, express in terms of k :

- b $\tan x$

- c $\tan 2x$

a) $\sin x \cos y + \cos x \sin y = 5 \sin x \cos y - 5 \cos x \sin y$
 $\frac{1}{2} + \frac{1}{3} = 5 \times \frac{1}{2} - 5 \times \frac{1}{3} \Rightarrow \frac{5}{6} = \frac{5}{6} \quad \text{true}$

b) $\sin x \cos y = \frac{1}{2} \quad \cos x \sin y = \frac{1}{3}$

$$\frac{\sin x \cos y}{\cos x \sin y} = \frac{\frac{1}{2}}{\frac{1}{3}} \Rightarrow \frac{\tan x}{\tan y} = \frac{3}{2} \quad \tan x = \frac{3}{2} k$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \quad \tan x = \frac{3}{2} \quad \text{h}$$

14 a Prove, by counter-example, that the statement $\sec(A+B) \equiv \sec A + \sec B$, for all A and B is false.

(2 marks)

b Prove that $\tan \theta + \cot \theta \equiv 2 \operatorname{cosec} 2\theta$, $\theta \neq \frac{n\pi}{2}$, $n \in \mathbb{Z}$.

(4 marks)

$$a) \sec(A+B) = \frac{1}{\cos(A+B)} \quad \frac{1}{\cos(A+B)} = \frac{1}{\cos A} + \frac{1}{\cos B}$$

$$\sec A + \sec B = \frac{1}{\cos A} + \frac{1}{\cos B}$$

$$A = 60 \quad \frac{1}{\cos(60+30)} = \frac{1}{\cos 60} + \frac{1}{\cos 30}$$

$$B = 30$$

$$0 \neq 2 + \frac{2}{\sqrt{3}}$$

b Prove that $\tan \theta + \cot \theta \equiv 2 \operatorname{cosec} 2\theta$, $\theta \neq \frac{n\pi}{2}$, $n \in \mathbb{Z}$.

$$\text{LHS: } \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} = \frac{1}{\frac{1}{2} \sin 2\theta} = \frac{2}{\sin 2\theta} = 2 \operatorname{cosec} 2\theta$$

19 a Given that $\sin^2 \frac{\theta}{2} = 2 \sin \theta$, show that $\sqrt{17} \sin(\theta + \alpha) = 1$ and state the value of α , where $0 \leq \alpha \leq \frac{\pi}{2}$.

(3 marks)

b Hence, or otherwise, solve $\sin^2 \frac{\theta}{2} = 2 \sin \theta$ for $0 \leq \theta \leq 360^\circ$.

(4 marks)

$$a) \sqrt{17} \sin(\theta + \alpha) = \sqrt{17} \sin \theta \cos \alpha + \sqrt{17} \cos \theta \sin \alpha$$

$$\cos 2x = (1 - 2 \sin^2 x) \Rightarrow \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$\text{where } x = \frac{\theta}{2}$$

$$\sin^2 \left(\frac{\theta}{2} \right) = \frac{1}{2} - \frac{1}{2} \cos \theta$$

$$\frac{1}{2} - \frac{1}{2} \cos \theta = 2 \sin \theta \Rightarrow$$

$$1 - \cos \theta = 4 \sin \theta \Rightarrow 4 \sin \theta + \cos \theta = 1$$

$$R \sin(\theta + \alpha) = R \sin \theta \cos \alpha + R \cos \theta \sin \alpha$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{1}{4} \quad \tan \alpha = \frac{1}{4} \quad R = \sqrt{4^2 + 1^2} = \sqrt{17}$$

$$\sqrt{17} \sin(\theta + 14.04) = 1$$

$$b) \sin^2\left(\frac{\theta}{2}\right) = 2 \sin \theta$$

$$\sqrt{17} \sin(0.4404) = 1$$

22 a Use the double-angle formulae to prove that $\frac{1 - \cos 2x}{1 + \cos 2x} \equiv \tan^2 x$.

$$\frac{1 - (1 - 2\sin^2 x)}{1 + 2\cos^2 x - 1} = \frac{2\sin^2 x}{2\cos^2 x} = \tan^2 x$$

23 a Prove that $\cos^4 2\theta - \sin^4 2\theta \equiv \cos 4\theta$. (4 marks)

b Hence find, for $0 \leq \theta \leq 180^\circ$, all the solutions of $\cos^4 2\theta - \sin^4 2\theta = \frac{1}{2}$ (2 marks)

$$a) (\cos^2 2\theta + \sin^2 2\theta) (\cos^2 2\theta - \sin^2 2\theta) \\ 1 \times \cos(2 \times 2\theta) = \cos 4\theta$$

$$b) \cos 4\theta = \frac{1}{2}, 4\theta = 60, 300, 360, 600$$

27 a Express $12 \sin x + 5 \cos x$ in the form $R \sin(x + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < 90^\circ$. Round α to 1 decimal place. (4 marks)

A runner's speed, v in m/s, in an endurance race can be modelled by the equation

$$v(x) = \frac{50}{12 \sin\left(\frac{2x}{5}\right) + 5 \cos\left(\frac{2x}{5}\right)}, 0 \leq x \leq 300$$

where x is the time in minutes since the beginning of the race.

b Find the minimum value of v . (2 marks)

c Find the time into the race when this speed occurs. (1 mark)

$$a) R \cos \alpha = 12 \quad R \sin \alpha = 5 \quad k = \sqrt{12^2 + 5^2} \\ R = 13$$

$$\tan \alpha = \frac{5}{12} \Rightarrow \alpha = 22.6^\circ (1 \text{ d.p.})$$

$$b) v(x) = \frac{50}{13 \sin\left(\frac{2x}{5} + 22.6\right)}$$

$$\text{min } v(x) = \frac{50}{13} =$$

$$c) \sin\left(\frac{2x}{5} + 22.6\right) = 1 \quad \frac{2x}{5} + 22.6 = 90, 270$$

1 Prove the identities:

a $\frac{\cos 2\theta + \cos 4\theta}{\sin 2\theta - \sin 4\theta} \equiv -\cot \theta$

$\cos A + \cos B$

$\sin A - \sin B$

$$\frac{\cos^2 \theta - \sin^2 \theta + \cos^2 2\theta - \sin^2 2\theta}{2 \sin \theta \cos \theta - 2 \sin 2\theta \cos 2\theta}$$

$$\frac{\cos^2 \theta - \sin^2 \theta + (\cos^2 \theta - \sin^2 \theta)^2 - (2 \sin \theta \cos \theta)^2}{2 \sin \theta \cos \theta - 2 \sin \theta \cos \theta \cos 2\theta}$$

$$\frac{\cos^2 \theta - \sin^2 \theta + \cos^4 \theta - 2 \cos^2 \theta \sin^2 \theta + \sin^4 \theta - 4 \sin^2 \theta \cos^2 \theta}{2 \sin \theta \cos \theta (1 - \cos 2\theta)}$$

$$\frac{\cos^2 \theta - \sin^2 \theta + \cos^4 \theta + \sin^4 \theta - 6 \cos^2 \theta \sin^2 \theta}{2 \sin \theta \cos \theta (1 - \cos 2\theta)}$$

numerator

$$\cos^2 \theta (1 + \cos^2 \theta - 6 \sin^2 \theta) - \sin^2 \theta (1 - \sin^2 \theta) = \cos^2 \theta (1 + \cos^2 \theta - 6 \sin^2 \theta) - \sin^2 \theta (\cos^2 \theta)$$

$$= \cos^2 \theta (1 + \cos^2 \theta - 6 \sin^2 \theta - \sin^2 \theta)$$

$$= \cos^2 \theta (\cos^2 \theta + \cancel{\sin^2 \theta} + \cos^2 \theta - 6 \sin^2 \theta - \cancel{\sin^2 \theta})$$

$$= \cos^2 \theta (2 \cos^2 \theta - 6 \sin^2 \theta) \rightarrow \cos^2 \theta (2 \cos^2 \theta - 6 + 6 \cos^2 \theta) = \cos^2 \theta (8 \cos^2 \theta - 6)$$

Denominator: $2 \sin \theta \cos \theta (1 - 2 \cos^2 \theta + 1)$

$$2 \sin \theta \cos \theta (2 - 2 \cos^2 \theta) = 4 \sin \theta \cos \theta (1 - \cos^2 \theta) \\ = 4 \sin^2 \theta \cos \theta$$

1 a Write $\cos 2\theta$ as $\cos(3\theta - \theta)$ and write $\cos 4\theta$ as $\cos(3\theta + \theta)$.

Then, using $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$,

$$\cos 2\theta = \cos 3\theta \cos \theta + \sin 3\theta \sin \theta$$

$$\cos 4\theta = \cos 3\theta \cos \theta - \sin 3\theta \sin \theta$$

$$\Rightarrow \cos 2\theta + \cos 4\theta = 2 \cos 3\theta \cos \theta$$

Similarly, using $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$,

$$\sin 2\theta = \sin 3\theta \cos \theta - \cos 3\theta \sin \theta$$

$$\sin 4\theta = \sin 3\theta \cos \theta + \cos 3\theta \sin \theta$$

$$\Rightarrow \sin 2\theta - \sin 4\theta = -2 \cos 3\theta \sin \theta$$

Therefore,

$$\frac{\cos 2\theta + \cos 4\theta}{\sin 2\theta - \sin 4\theta} = \frac{2 \cos 3\theta \cos \theta}{-2 \cos 3\theta \sin \theta} \\ = -\frac{\cos \theta}{\sin \theta} \\ = -\cot \theta \quad \text{as required.}$$

$$\frac{2 \cos\left(\frac{6\theta}{2}\right) \cos\left(\frac{2\theta}{2}\right)}{2 \cos\left(\frac{6\theta}{2}\right) \sin\left(\frac{-2\theta}{2}\right)} \\ = \frac{2 \cos 3\theta \cos \theta}{2 \cos 3\theta \sin(-\theta)} \\ = \frac{\cos \theta}{\sin(-\theta)} = -\cot \theta$$