

Example 1

Find the following integrals.

a $\int \left(2 \cos x + \frac{3}{x} - \sqrt{x} \right) dx$

b $\int \left(\frac{\cos x}{\sin^2 x} - 2e^x \right) dx$

a) $2 \int \cos x dx + 3 \int \frac{1}{x} dx - \int x^{1/2} dx$
 $= 2 \sin x + 3 \ln x - \frac{2}{3} x^{3/2} + C$

b) $\int \frac{1}{\sin x} \times \frac{\cos x}{\sin x} dx - 2 \int e^x dx$

$\int \operatorname{cosec} x \cot x dx - 2 \int e^x dx = -\operatorname{cosec} x - 2e^x + C$

g $\int \operatorname{cosec}^2 x (1 + \tan^2 x) dx$

h $\int \sec^2 x (1 - \cot^2 x) dx$

i $\int \sec^2 x (1 + e^x \cos^2 x) dx$

j $\int \left(\frac{1 + \sin x}{\cos^2 x} + \cos^2 x \sec x \right) dx$

g) $\int (\operatorname{cosec}^2 x + \operatorname{cosec}^2 x \tan^2 x) dx$
 $= \cot x + \int \frac{1}{\sin^2 x} \times \frac{\sin^2 x}{\cos^2 x} dx$
 $= \cot x + \int \sec^2 x dx$
 $= \cot x + \tan x + C$

j) $\int \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos x} + \cos^2 x \sec x \right) dx$
 $\int (\sec^2 x + \sec x \tan x + \cos x) dx$
 $= \tan x + \sec x + \sin x + C$

Example 3

Find the following integrals.

a $\int \cos(2x + 3) dx$

b $\int e^{4x+1} dx$

c $\int \sec^2 3x dx$

a) Assume $u = \sin(2x+3)$

b) $\int e^{4x+1} dx = \frac{e^{4x+1}}{(4x+1)'} + C$
 $= \frac{1}{4} e^{4x+1} + C$

$\frac{dI}{dx} = 2 \cos(2x+3) \rightarrow 2x \text{ initial}$

$\int \cos(2x+3) dx = \frac{1}{2} \sin(2x+3) + C$

c) $\int \sec^2 3x dx = \frac{\tan x}{(3x)'} + C = \frac{1}{3} \tan 3x + C$

Example**4**

Find the following integrals:

a $\int \frac{1}{3x+2} dx$

b $\int (2x+3)^4 dx$

a) $\frac{\ln(3x+2)}{3} + c$

b) $\frac{1}{5} \frac{(2x+3)^5}{(2x+3)'} + c$
 $= \frac{1}{10} (2x+3)^5$

8 Given $\int_{\frac{\pi}{4k}}^{\frac{\pi}{3k}} (1 - \pi \sin kx) dx = \pi(7 - 6\sqrt{2})$,

find the exact value of k .

(7 marks)

$$\left[x + \frac{\pi}{k} \cos kx \right]_{\frac{\pi}{4k}}^{\frac{\pi}{3k}} = \pi(7 - 6\sqrt{2})$$

$$\left(\frac{\pi}{3k} + \frac{\pi}{k} \cos\left(k \times \frac{\pi}{3k}\right) \right) - \left(\frac{\pi}{4k} + \frac{\pi}{k} \cos\left(k \times \frac{\pi}{4k}\right) \right) = \pi(7 - 6\sqrt{2})$$

$$\left(\frac{\pi}{3k} + \frac{\pi}{k} \times \frac{1}{2} \right) - \left(\frac{\pi}{4k} + \frac{\pi}{k} \times \frac{\sqrt{2}}{2} \right) = \pi(7 - 6\sqrt{2})$$

$$\frac{5\pi}{6k} - \frac{(1 + \frac{1}{2}\sqrt{2})\pi}{4k} = \pi(7 - 6\sqrt{2})$$

$$\left(\frac{7 - 6\sqrt{2}}{12k} \right) \pi = \pi(7 - 6\sqrt{2}) \Rightarrow \frac{1}{12k} = 1 \quad \left(k = \frac{1}{12} \right)$$

Example**5**Find $\int \tan^2 x dx$

We don't know the $\int \tan^2 x dx$
 but we know that $\int \sec^2 x dx = \tan x$

we also know that $\sec^2 x = \tan^2 x + 1$
 $\therefore \tan^2 x = \sec^2 x - 1$

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx$$

$$= \tan x - x + c$$

Example**6**

Show that $\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sin^2 x dx = \frac{\pi}{48} + \frac{1 - \sqrt{2}}{8}$

We don't know how to deal with $\int \sin^2 x dx$ or $\int \cos^2 x dx$
 when we see $\int \sin^2 x dx$ or $\int \cos^2 x dx$ we think of $\cos 2x$
 $\cos 2x = 1 - 2\sin^2 x \quad \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$

$$\int \sin^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) dx$$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x + C$$

Example 7

Find:

a $\int \sin 3x \cos 3x \, dx$

b $\int (\sec x + \tan x)^2 \, dx$

a) $\sin 2a = 2 \sin a \cos a$
 $\sin a \cos a = \frac{1}{2} \sin 2a$
 where $a = 3x$

$$\int \frac{1}{2} \sin 6x \, dx = -\frac{1}{2} \times \frac{1}{6} \cos 6x + C$$

$$= -\frac{1}{12} \cos 6x + C$$

b) $\int (\sec^2 x + 2 \sec x \tan x + \tan^2 x) \, dx$

$\xrightarrow{\sec^2 x - 1}$

$$= \int (\sec^2 x + \sec^2 x - 1) \, dx + 2 \int \sec x \tan x \, dx$$

$$= 2 \tan x - x + 2 \sec x + C$$

6 $f(x) = 5 \sin^2 x + 7 \cos^2 x$

a Show that $f(x) = \cos 2x + 6$. (3 marks)

b Hence, find the exact value of $\int_0^{\frac{\pi}{4}} f(x) \, dx$. (4 marks)

a) $\cos 2x = 1 - 2 \sin^2 x \Rightarrow \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$

$\cos 2x = 2 \cos^2 x - 1 \Rightarrow \cos^2 x = \frac{1}{2} \cos 2x + \frac{1}{2}$

$$f(x) = 5 \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) + 7 \left(\frac{1}{2} \cos 2x + \frac{1}{2} \right)$$

$$= \frac{5}{2} - \frac{5}{2} \cos 2x + \frac{7}{2} \cos 2x + \frac{7}{2}$$

$$= \cos 2x + 6$$

b) $\int (\cos 2x + 6) \, dx = \frac{1}{2} \sin 2x + 6x + C$

Example 8

Find

a $\int \frac{2x}{x^2 + 1} \, dx$

b $\int \frac{\cos x}{3 + 2 \sin x} \, dx$

a) we can see that the derivative of $(x^2 + 1)$ will give us the numerator

$\therefore$ we deal only with $\int \frac{1}{x^2 + 1} \, dx$

Assume $I = \ln |x^2 + 1|$

$$I = \ln|x^2+1|$$

go backwards $\frac{dy}{dx} = \frac{1}{x^2+1} \times (x^2+1)' = \frac{2x}{x^2+1}$ that's the original

$$\therefore \int \frac{2x}{x^2+1} dx = \ln|x^2+1| + c$$

b $\int \frac{\cos x}{3+2\sin x} dx$

We can see that $(3+2\sin x)' = \cos x$

deal only with $\int \frac{1}{3+2\sin x} dx$

Assume $I = \ln|3+2\sin x|$

go back $\frac{dy}{dx} = \frac{1}{3+2\sin x} (3+2\sin x)' = \frac{2\cos x}{3+2\sin x}$ (x2 original)

$$\int \frac{\cos x}{3+2\sin x} dx = \frac{1}{2} \ln|3+2\sin x| + c$$

Example 9

Find:

a $\int 3\cos x \sin^2 x dx$

b $\int x(x^2+5)^3 dx$

a) deal only with $\int \sin^2 x dx$

Assume $I = \sin^3 x$

go back $\frac{dy}{dx} = 3\sin^2 x (\sin x)' = 3\cos x \sin^2 x$ (the original)

$$\therefore \int 3\cos x \sin^2 x dx = \sin^3 x + c$$

b $\int x(x^2+5)^3 dx$

deal with $\int (x^2+5)^3 dx$

Assume $I = (x^2+5)^4$ $\frac{dy}{dx} = 4(x^2+5)^3 \times (2x) = 8x(x^2+5)^3$ (x8 greater)

$$\int x(x^2+5)^3 dx = \frac{1}{8} (x^2+5)^4 + c$$

Example 10

Use integration to find $\int \frac{\operatorname{cosec}^2 x}{(2 + \cot x)^3} dx$

deal with $\int (2 + \cot x)^{-3} dx$

$$\begin{aligned} \text{Assume } I &= (2 + \cot x)^{-2} \quad \frac{dy}{dx} = -2(2 + \cot x)^{-3} \times (2 + \cot x)' \\ &= \frac{-2 \operatorname{cosec}^2 x}{(2 + \cot x)^3} \quad (\times (-1) \text{ the original}) \end{aligned}$$

$$\int \frac{\operatorname{cosec}^2 x}{(2 + \cot x)^3} dx = -\frac{1}{2} (2 + \cot x)^{-2} + c$$

Example 11

Given that $\int_0^\theta 5 \tan x \sec^4 x dx = \frac{15}{4}$ where $0 < \theta < \frac{\pi}{2}$, find the exact value of θ .

deal only with $\int \sec^4 x dx$

$$\begin{aligned} \text{Assume } I &= \sec^5 x \quad \frac{dy}{dx} = 5 \sec^4 x (\sec x)' \\ &= 5 \sec^4 x \sec x \tan x \\ &= 5 \sec^5 x \tan x \quad (\text{the power of } \sec x \text{ is higher}) \end{aligned}$$

wrong assumption above
try again

$$\begin{aligned} \text{Assume } I &= \sec^4 x \quad \frac{dy}{dx} = 4 \sec^3 x (\sec x \tan x) \\ &= 4 \tan x \sec^4 x \end{aligned}$$

$$\therefore \int 5 \tan x \sec^4 x dx = \frac{5}{4} \sec^4 x + c$$

5 Given that $\int_0^\theta 4 \sin 2x \cos^4 2x dx = \frac{4}{5}$ where $0 < \theta < \pi$, find the exact value of θ .

deal with $\int \cos^4 2x dx$

$$\begin{aligned} \text{Assume } I &= \cos^5 2x \quad \frac{dy}{dx} = 5 \cos^4 2x (\cos 2x)' \\ &= -10 \sin 2x \cos^4 2x \quad (\times (-\frac{4}{10}) \text{ the original}) \end{aligned}$$

$$\begin{aligned} \int 4 \sin 2x \cos^4 2x dx &= -\frac{4}{10} \cos^5 2x + c \\ &= -\frac{2}{5} \cos^5 2x + c \end{aligned}$$

6 a By writing $\cot x = \frac{\cos x}{\sin x}$, find $\int \cot x \, dx$.

b Show that $\int \tan x \, dx \equiv \ln|\sec x| + c$.

a) $\int \frac{\cos x}{\sin x} \, dx$

Let with $\int \frac{1}{\sin x} \, dx$

As since $\Rightarrow I = \ln|\sin x|$ $\frac{dy}{dx} = \frac{1}{\sin x} \times (\sin x)'$
 $= \frac{\cos x}{\sin x}$ (original)

$\int \cot x \, dx = \ln|\sin x| + c$

Example 12

Find $\int x\sqrt{2x+5} \, dx$ using the substitutions:

a $u = 2x+5$

b $u^2 = 2x+5$

a) $u = 2x+5$
 $\downarrow$
 $\frac{du}{dx} = 2$
 $dx = \frac{1}{2} du$

$\int x u^{1/2} \, dx = \int x u^{1/2} \frac{1}{2} du$
 $x = \frac{1}{2}u - \frac{5}{2}$
 $= \int \left(\frac{1}{2}u - \frac{5}{2}\right) u^{1/2} \frac{1}{2} du$
 $= \int \left(\frac{1}{4} u^{3/2} - \frac{5}{4} u^{1/2}\right) du$

6 a By writing $\cot x = \frac{\cos x}{\sin x}$, find $\int \cot x \, dx$.

Let's use substn. for the previous quest.

$\int \frac{\cos x}{\sin x} \, dx$ $u = \sin x \Rightarrow \frac{du}{dx} = \cos x$ $dx = \frac{1}{\cos x} du$

$\downarrow$
 $\int \frac{\cancel{\cos x}}{u} \times \frac{1}{\cancel{\cos x}} du = \int \frac{1}{u} du = \ln u + c$
 $= \ln|\sin x| + c$

Given that $\int_0^\theta 4 \sin 2x \cos^4 2x \, dx = \frac{4}{5}$ where $0 < \theta < \pi$, find the exact value of θ .

$$u = \cos 2x \quad \frac{du}{dx} = -2 \sin 2x \quad dx = \frac{-1}{2 \sin 2x} du$$

$$\int \cancel{u \sin 2x} u^4 \left(\cancel{-\frac{1}{2 \sin 2x}} \right) du = \int -2 u^4 du$$

$$= -\frac{2}{5} u^5 + c$$

$$= -\frac{2}{5} \cos^5 2x + c$$

Example 13

Use the substitution $u = \boxed{}$ to find

$$\int \cos x \sin x (1 + \sin x)^3 \, dx$$

$$u = 1 + \sin x \quad \frac{du}{dx} = \cos x \quad dx = \frac{1}{\cos x} du$$

$$\int \cancel{\cos x} \sin x (u)^3 \frac{1}{\cancel{\cos x}} du = \int (u-1) u^3 du$$

$$\begin{array}{c} ? \\ \downarrow \\ u-1 \end{array}$$

$$= \int u^4 - u^3 du = \frac{1}{5} u^5 - \frac{1}{4} u^4 + c$$

$$= \frac{1}{5} (1 + \sin x)^5 - \frac{1}{4} (1 + \sin x)^4 + c$$