

Integration

11.1 Integrating standard functions

- 1 a $e^x - 2 \cos x + c$ b $4 \sin x + \frac{3}{x} + c$
 c $5 \ln|x| - 2e^x + c$ d $\frac{2}{3}x^{\frac{3}{2}} - \frac{3}{4}\cos x + c$
 e $\frac{11}{3}x^3 + \frac{1}{2}e^x - 6 \ln|x| + c$
 f $3 \sin x - 2\sqrt{x} + c$
- 2 a $-\cot x + c$ b $\sec x + c$
 c $-\operatorname{cosec} x + c$ d $4 \tan x + c$
 e $-4 \cot x + c$ f $8 \operatorname{cosec} x + c$
- 3 a $\tan x + \sec x + c$ b $\frac{7}{2}x^2 - 5 \ln x + c$
 c $3e^x + 5 \cot x + c$ d $\frac{1}{2}x^2 - x + \ln x + c$
- 4 a 15 b $\ln \frac{5}{2} - 117$ c $4e^{\frac{\pi}{3}} - 3$
 d $5 - \sqrt{2}$ e 20
- 5 $5 + \ln \frac{9}{4}$
- 6 $p = 4$
- 7 $3e^{\frac{\pi}{6}} - 3e^{\frac{\pi}{3}} = 3(e^{\frac{\pi}{6}} - e^{\frac{\pi}{3}})$
- 8 $[2x + 3 \ln x]_{2a}^{5a} = 6a + 3 \ln \frac{5}{2}$
 $\Rightarrow 6a + 3 \ln \frac{5}{2} = \ln 1000 = 3 \ln 10$
 $\Rightarrow 6a = 3 \ln 4$
 $\Rightarrow a = \frac{1}{2} \ln 4 = \ln 2$

11.2 Integrating $f(ax+b)$

- 1 a i $16(4x-5)^3$ ii $-12(3-2x)^5$
 b i $(4x-5)^4 + c$ ii $\frac{1}{16}(4x-5)^4 + c$
 iii $(3-2x)^6 + c$ iv $-\frac{1}{6}(3-2x)^6 + c$
- 2 a $2e^{3x} + c$ b $\frac{1}{3}\sin(3x+2) + c$
 c $\frac{1}{24}(4x+1)^6 + c$ d $-\frac{1}{4}\operatorname{cosec} 4x + c$
 e $\frac{1}{8}\tan(8x-1) + c$ f $\frac{1}{5}\cot(1-5x) + c$
 g $\frac{5}{2}\ln|2x-3| + c$ h $-11e^{4-x} + c$
- 3 a $-\frac{1}{4(4x-3)} - \frac{1}{12}(4x-3)^3 + c$
 b $\frac{1}{10}e^{5x} - \frac{2}{3}\tan 3x + c$
 c $\frac{1}{4}e^{4x} - 2x - \frac{1}{4}e^{-4x} + c$
 d $-\frac{5}{2}\cot 2x + \frac{3}{2}\operatorname{cosec} 2x + c$
- 4 a $\ln \frac{7}{3}$ b $\sqrt{2} - 1$
 c $16 \ln 4 - \frac{33}{2}$ d $1 - \arctan\left(\frac{\pi}{8}\right)$
- 5 4
- 6 $2\frac{\ln 25}{\ln 7}$
- 7 $54 + \ln 256$
- 8 $k = -\frac{1}{9}$

11.3 Using trigonometric identities

- 1 a $\frac{1}{2}x - \frac{1}{4}\sin 2x + c$
 b $-3 \cot x - 3x + c$
 c $\frac{3}{2}x + 2 \sin x + \frac{1}{4}\sin 2x + c$
 d $\tan x - x + c$
- 2 a $\sqrt{3} - 1 - \frac{\pi}{6}$ b $\frac{10-3\pi}{4}$
 c $\frac{4\sqrt{3}-\pi}{3}$ d $4 - 2\sqrt{2} - \frac{\pi}{4}$

- 3 a i $\cos 5x \cos 2x - \sin 5x \sin 2x$
 ii $\cos 5x \cos 2x + \sin 5x \sin 2x$
 b L.H.S. = $\cos 7x + \cos 3x$
 R.H.S. = $2 \cos 5x \cos 2x$
 c $\frac{2}{7}\sin 7x + \frac{2}{3}\sin 3x + c$

$$4 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 x \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{2} \cos 2x + \frac{1}{2} \right) dx$$

$$= \left[\frac{1}{4} \sin 2x + \frac{1}{2} x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(0 - \frac{\pi}{4} \right) - \left(\frac{\sqrt{3}}{2} - \frac{\pi}{12} \right)$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

- 5 Use the trigonometric identity $\tan^2 2x + 1 = \sec^2 2x$ to obtain $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} (\sec^2 2x - 1) \, dx = \left[\frac{1}{2} \tan 2x - x \right]_{\frac{\pi}{12}}^{\frac{\pi}{6}}$
 $= \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) - \left(\frac{1}{2\sqrt{3}} - \frac{\pi}{12} \right)$
 $= \frac{\sqrt{3}}{3} - \frac{\pi}{12}$
- 6 a $\sin 6x \cos x + \cos 6x \sin x = \sin 7x$ (1)
 $\sin 6x \cos x - \cos 6x \sin x = \sin 5x$ (2)
 Adding (1) and (2): $2 \sin 6x \cos x = \sin 7x + \sin 5x$
 b $-\frac{3}{14} \cos 7x - \frac{3}{10} \cos 5x + c$

11.4 Reverse chain rule

- 1 a i $6x(x^2-4)^2$ ii $8(x+2)(x^2+4x-5)^3$
 b i $(x^2-4)^3 + c$ ii $\frac{5}{2}(x^2-4)^3 + c$
 iii $-\frac{4}{3}(x^2+4x-5)^3 + c$
- 2 a $2(x^2-7)^{\frac{3}{2}} + c$ b $-\frac{5}{2}e^{\cos 2x} + c$
 c $\frac{1}{5}(4x^3+16x-5)^{\frac{5}{2}} + c$
 d $\frac{2}{3}(1+\tan x)^{\frac{3}{2}} + c$
 e $-\frac{1}{3}e^{\cot 3x} + c$ f $-\frac{1}{25}\cos^5 5x + c$
- 3 a $-2 \ln|2-x^2| + c$ b $\frac{1}{3}\ln|4-3\cos x| + c$
 c $-\frac{1}{2}\ln|14+9x-6x^3| + c$
 d $-2 \ln|\cos x| + c$
 e $\frac{1}{8}\ln|4-\cos 2x| + c$
 f $-\frac{1}{3}\ln|1+e^{-3x}| + c$
- 4 a $\frac{1}{12}$ b $\frac{1}{3}\ln 2$
 c $2 \ln \frac{28}{9}$ d $\frac{1}{3}(e - e^{\sqrt{2}})$
- 5 $3 + 4\sqrt{3}$
- 6 a $\sin^5 x \equiv \sin x(\sin^2 x)(\sin^2 x)$
 $\equiv \sin x(1 - \cos^2 x)(1 - \cos^2 x)$
 $\equiv \sin x(1 - 2\cos^2 x + \cos^4 x)$
 $\equiv \sin x - 2 \sin x \cos^2 x + \sin x \cos^4 x$
 b $-\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + c$
- 7 $32\sqrt{3}$
- 8 $\frac{1}{4}(e-1)$
- 9 $\frac{19}{648}$
- 10 $k = \sqrt{6}$

11.5 Integration by substitution

- 1 a 1 b $2(u-2)$
 c Substitute $2x = 2(u-2)$, $(x+2)^4 = u^4$ and $dx = du$
 d $\int 2(u-2)u^4 du = \int (2u^5 - 4u^4) du = \frac{1}{3}u^6 - \frac{4}{5}u^5 + c$
 e $\frac{1}{3}(x+2)^6 - \frac{4}{5}(x+2)^5 + c$
- 2 a $\frac{1}{6}(x-8)^6 + \frac{8}{5}(x-8)^5 + c$
 b $-\frac{1}{7}(1+\cos x)^7 + \frac{1}{6}(1+\cos x)^6 + c$
 c $-\frac{2}{5}(2+\cot x)^{\frac{5}{2}} + \frac{4}{3}(2+\cot x)^{\frac{3}{2}} + c$
 d $\sqrt{x^2+1} + c$
- 3 a $\frac{16}{15}$ b $8\ln\frac{4}{5}$
 c -1 d $2 - \ln 4$
- 4 $\frac{2}{15} + \frac{10\sqrt{5}}{3}$
- 5 $\ln\frac{7}{3}$
- 6 a $x = \tan \theta$, $\frac{dx}{d\theta} = \sec^2 \theta$

$$\int \frac{1}{1+x^2} dx = \int \frac{1}{1+\tan^2 \theta} \sec^2 \theta d\theta$$

$$= \int \frac{1}{\sec^2 \theta} \sec^2 \theta d\theta$$

$$= \int 1 d\theta$$

$$= \theta + c$$

$$= \arctan x + c$$

- b $\frac{\pi}{12}$
- 7 $\int_0^4 \frac{5x}{\sqrt{2x+1}} dx = \int_1^3 \frac{5}{4} \left(\frac{u^2-1}{2} \right) u du = \frac{5}{2} \left[\frac{u^3}{3} - u \right]_1^3 = \frac{50}{3}$
- 8 $4 \sin x - 4 \ln|1 + \sin x| + c$

11.6 Integration by parts

- 1 a $x \sin x + \cos x + c$
 b $\ln|\cos x| + x \tan x + c$
 c $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$
 d $\frac{1}{16} \ln|\sin 4x| - \frac{1}{4}x \cot 4x + c$
 e $-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + c$
 f $-\frac{1}{5}x \cos 5x + \frac{1}{25} \sin 5x + c$
- 2 a $\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c$ b $\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + c$
 c $x \ln x - x + c$
- 3 a $-\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x + \frac{1}{4} \cos 2x + c$
 b $-\frac{1}{27}(9x^2 + 6x + 2)e^{-3x} + c$
 c $\frac{1}{2}x^2(1+x)^4 - \frac{1}{5}x(1+x)^5 + \frac{1}{30}(1+x)^6 + c$
- 4 a $\frac{\sqrt{2}}{8}(4-\pi)$ b $\frac{78}{5}$ c π
 d $\frac{\pi\sqrt{3}}{12} - \frac{1}{4} \ln 2$ e $-\frac{\pi}{32}$ f $6 - \frac{78}{e^3}$
- 5 $12 \ln 8 - \frac{3}{4} \ln 2 - \frac{45}{8}$
- 6 $\left[\frac{1}{4}(2x^2 - 2x + 1)e^{2x} \right]_1^2$
 $= \frac{1}{4}(5)e^4 - \frac{1}{4}(1)e^2$
 $= \frac{e^2}{4}(5e^2 - 1)$
- 7 a $-\frac{1}{\pi}x \cos \pi x + \frac{1}{\pi^2} \sin \pi x + c$
 b $\frac{17\pi^2 - 8}{2\pi^3}$

8 $\frac{\pi}{2} - \frac{\pi^2}{8} + \ln 2$

- 9 a $\frac{1}{2}x \sin 2x + \frac{1}{4} \cos 2x + c$
 b $\frac{1}{4}x^2 - \frac{1}{4}x \sin 2x - \frac{1}{8} \cos 2x + c$

11.7 Partial fractions

- 1 a $A = \frac{10}{3}$, $B = \frac{14}{3}$
 b $\frac{10}{3} \ln|x+2| + \frac{14}{3} \ln|x-4| + c$
- 2 a $\ln|x+5| + 3 \ln|x+2| + c$
 b $-4 \ln|x-2| - 3 \ln|x+3| + c$
 c $6 \ln|x-1| - 5 \ln|x+2| + c$
 d $\frac{6}{7} \ln|x-4| - \frac{6}{7} \ln|2x-1| + c$
- 3 a $(x+4)(2x-1) = 2x^2 + 7x - 4$

$$2x^2 + 7x - 4 \begin{array}{r} -2x^2 \\ \hline 7x - 4 \\ -2x^2 - 7x + 4 \\ \hline 7x + 10 \end{array}$$

 So $f(x) = \frac{7x+10}{(x+4)(2x-1)} - 1$
 b $A = 2$, $B = 3$
 c $2 \ln|x+4| + \frac{3}{2} \ln|2x-1| - x + c$
- 4 a Equate coefficients in
 $16x^2 - 37x + 17 = A(2x-3)^2 + B(4-x)(2x-3) + C(4-x)$
 to get simultaneous equations:
 $4A - 2B = 16$
 $-12A + 11B - C = -37$
 $9A - 12B + 4C = 17$
 Then solve to find $A = 5$, $B = 2$ and $C = -1$.
 b $-5 \ln|4-x| + \ln|2x-3| + \frac{1}{2(2x-3)} + c$
- 5 a $\frac{6}{x-1} - \frac{3}{x+2}$ b $\ln \frac{4096}{125}$
- 6 a $A = 1$, $B = -2$, $C = 1$ b $2 + \ln \frac{5}{3}$
- 7 a $\frac{12}{5(2x+1)} - \frac{6}{5(x+3)}$ b $\frac{6}{5} \ln \frac{4}{3}$
- 8 a $A = 1$, $B = -\frac{5}{2}$, $C = \frac{5}{2}$ b $2 + \frac{5}{2} \ln \frac{9}{5}$

11.8 Finding areas

- 1 $\frac{8}{5}$
- 2 a $x \ln \frac{1}{2}x = 0 \Rightarrow x = 0$ or $\ln \frac{1}{2}x = 0$
 So A is where $\ln \frac{1}{2}x = 0$
 $\Rightarrow \frac{1}{2}x = 1 \Rightarrow x = 2$
 So A has coordinates $(2, 0)$.
 b $R_1 = -\int_1^2 x \ln \left(\frac{1}{2}x \right) dx = -\left[\frac{1}{2}x^2 \ln \left(\frac{1}{2}x \right) - \frac{1}{4}x^2 \right]_1^2 = \frac{3}{4} - \frac{1}{2} \ln 2$
 c $R_2 = \int_2^3 x \ln \left(\frac{1}{2}x \right) dx = \left[\frac{1}{2}x^2 \ln \left(\frac{1}{2}x \right) - \frac{1}{4}x^2 \right]_2^3 = \frac{9}{2} \ln \frac{3}{2} - \frac{5}{4}$
 Total area $= R_1 + R_2 = \frac{9}{2} \ln \frac{3}{2} - \frac{1}{2} \ln 2 - \frac{1}{2}$
- 3 a $f(\pi) = 2 \cos \frac{1}{2}\pi + 1 = 1$
 $g(\pi) = -\frac{\pi}{\pi} + 2 = 1$
 b $4 - \frac{\pi}{2}$
- 4 a $2t$ b $t = 0$
 c $t \neq -2$ because y is positive and $2(-2) + 1 = -3$
 d $\frac{44}{3}$
- 5 a $\frac{1}{2} \left(\frac{\pi}{2} \right) \cos \left(\frac{\pi}{2} \right) = 0$, $\frac{1}{2} \left(\frac{3\pi}{2} \right) \cos \left(\frac{3\pi}{2} \right) = 0$
 b $\frac{5\pi}{4} - \frac{1}{2}$

6 a $(\pi^2, 0)$ b 8π

7 a $\sin x = -\cos 2x + 1$
 $\Rightarrow \sin x = -(1 - 2\sin^2 x) + 1$
 $\Rightarrow 2\sin^2 x - \sin x = 0$
 $\Rightarrow \sin x(2\sin x - 1) = 0$

$2\sin x = 1 \Rightarrow x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$
 So the x-coordinate of B is $\frac{5\pi}{6}$

b $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

8 $15\pi - 30$

11.9 The trapezium rule

1 a

x	0	0.5	1	1.5	2
y	0	0.375	1	1.125	0

b There are 4 trapezia and each has a height of 0.5 as
 $4 \times 0.5 = 2 = b - a$

c 1.25

d $\frac{4}{3}$

2 a

x	0	0.25	0.5	0.75	1
y	1	0.6116	0.6103	0.7435	0.9183

b 0.731

c $\frac{43}{60} - \frac{1}{4e^4}$

d The graph is convex, so the lines connecting the two endpoints would be above the curve, giving a larger answer than the real answer.

e Decrease the width of each strip by increasing the number of values used.

3 a

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y	0	0.142	0.436	0.531	0

b 0.436 c $\frac{\pi^2}{4} - 2$ d 6.8%

4 a

x	0	0.5	1	1.5	2	2.5	3
y	1.25	1.038	0.933	0.9	0.929	1.041	1.333

b 3.07 c $\ln(12\sqrt{3})$ d 1.2%

5 a

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y	0	0.1155	0.1768	0.0478	0

b 0.1336 c $\frac{2}{15}$

6 a

x	1	1.25	1.5	1.75	2
y	1	1.0872	1.2281	1.4285	1.6931

b 1.273 (3 d.p.)

c $\int \left(\frac{1}{4}x^2 \ln x + 1\right) dx = \frac{1}{12}x^3 \ln x - \frac{1}{36}x^3 + x + c$

Substitute in the limits and simplify to obtain $\frac{2}{3}\ln 2 + \frac{29}{36}$

11.10 Solving differential equations

1 a $\frac{dy}{dx} = \frac{x}{y} \times (x^2 + 6)^5 \Rightarrow \int y dy = \int x(x^2 + 6)^5 dx$

b Using the substitution $u = x^2 + 6$,

$\int y dy = \frac{1}{2} \int u^5 du$

$\Rightarrow \frac{1}{2}y^2 = \frac{1}{12}u^6 + c = \frac{1}{12}(x^2 + 6)^6 + c$

2 a $y = \frac{4x}{3} + c$

b $y = \arcsin\left(\frac{1}{2}x - \frac{1}{4}\sin 2x + c\right)$

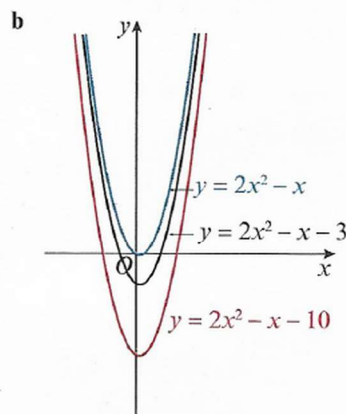
c $y = Ae^{2x^3+3x^2}$ d $y = -\frac{1}{2}\ln\left|\frac{2}{3}\cos^3 x + c\right|$

3 a $y = \frac{3(x+1)^2}{4(x+2)}$ b $y = \operatorname{arccot}(3 + \ln 2 - e^x - x)$

c $\tan x + \cot y - 2 = 0$

d $x^2 + y^2 = 13^2$

4 a $y = 2x^2 - x + c$



5 $y^2 = 1 - \frac{1}{x^2}e^{-x^2+1}$

6 $\sec y = 2x - \frac{1}{2}\sin 2x + \sqrt{2} + \frac{1}{2} - \frac{\pi}{2}$

7 $\ln|e^{2y} - 4| = 2e^x(x-1) + c$

8 a $\frac{2}{2x-5} + \frac{3}{x+2}$

b $y = A(2x-5)(x+2)^3$

c $y = \frac{1}{5}(2x-5)(x+2)^3$

11.11 Modelling with differential equations

1 a Separate the variables: $\int P^{-\frac{1}{3}} dP = \int 3 dt$ and integrate to obtain $\frac{3}{2}P^{\frac{2}{3}} = 3t + c$.

b $c = 150$

c $\frac{3}{2}P^{\frac{2}{3}} = 3t + 150 \Rightarrow P^{\frac{2}{3}} = 2t + 100 \Rightarrow P = (2t + 100)^{\frac{3}{2}}$

d 42.1 years

2 a i The rate of change of the value of the laptop with respect to time.

ii The value of the laptop is decreasing

iii Proportional to the existing value of the laptop.

b $\int \frac{1}{V} dV = \int -k dt \Rightarrow \ln V = -kt + c \Rightarrow V = e^{-kt+c}$
 $\Rightarrow V = Ae^{-kt}$

When $t = 0$, $V = V_0 = Ae^{-k \times 0}$, so $A = V_0$ and $V = V_0 e^{-kt}$

c 800

d $k = 0.094$

3 a $\frac{dP}{dt} = kP \Rightarrow \int \frac{1}{P} dP = \int k dt$

$\Rightarrow \ln P = kt + c \Rightarrow P = e^{kt+c}$

$\Rightarrow P = Ae^{kt}$

$A = 1600$, $k = \frac{1}{50}\ln \frac{13}{8}$

b 2013

c The model is clearly inaccurate for larger values of t .

4 a $\frac{dT}{dt}$ is the rate of change of the temperature of the soup with respect to time.

$(T - 20)$ is the difference between the temperature of the soup and the room temperature.

$-k$ signifies the soup cooling.

b 15.1 minutes

$$5 \quad \mathbf{a} \quad \frac{dh}{dt} = \frac{100\pi - 3\pi h}{900\pi}$$

$$900 \frac{dh}{dt} = 100 - 3h$$

$$\mathbf{b} \quad 300 \ln 4 \text{ seconds}$$

$$6 \quad \mathbf{a} \quad \frac{dA}{dr} = 2\pi r$$

$$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$$

$$= \frac{1}{2\pi r} \times \left(-k \sin\left(\frac{t}{4\pi}\right) \right)$$

$$= -\frac{k}{2\pi r} \sin\left(\frac{t}{4\pi}\right)$$

$$\mathbf{b} \quad r^2 = 75 \cos\left(\frac{t}{4\pi}\right) + 25$$

$$\mathbf{c} \quad 24.0 \text{ minutes}$$

Problem solving: Set A

$$\mathbf{B} \quad \frac{1}{25} (e^5 - 6)$$

$$\mathbf{S} \quad \frac{86}{5}$$

$$\mathbf{G} \quad 16 - \frac{26}{3}\sqrt{3}$$

Problem solving: Set B

$$\mathbf{B} \quad \mathbf{a} \quad y = k(x^2 + x + 2)^2$$

$$\mathbf{b} \quad y = \frac{(x^2 + x + 2)^2}{4}$$

$$\mathbf{S} \quad \ln|e^{2y} - 1| = 2e^x(x - 1) + 2 + \ln 3$$

$$\mathbf{G} \quad \mathbf{a} \quad T = \frac{80000}{1 + 4e^{-\frac{2}{3}t}} \quad \mathbf{b} \quad 80000$$